SECTIONING ANGLES USING HYPERBOLIC CURVES

NICHOLAS MOLBERT, TIA BURDEN, AND JULIE FINK

Abstract. In the following paper, we will derive an equation of a hyperbola which passes through a point that, when connected to the foci of the hyperbola, creates a triangle such that measure of one base angle is twice of the other. Consequently, all points on this hyperbolic curve create triangles with this characteristic. Applying this to a specific condition, it is possible to show that with this hyperbola, a straight edge, and a compass, any angle can be trisected.

1. History of the Problem of Trisection

The problem of trisecting an angle goes back to the time of the Ancient Greeks. Plato was the first to think of constructions as a process that must only be done with a straightedge, used to connect two points, and a compass, used to create circles and arcs. Trisecting the angle is one of three problems of Greek antiquity, the other two being creating a square with identical area as a circle and constructing a cube twice the volume of a given cube.

Ultimately, the Greeks strived to trisect angles of arbitrary measure. They believed it was possible simply because some angles, such as 90 degrees, were shown to be easily trisected. Furthermore, processes had been discovered to both bisect and trisect line segments, so could any arbitrary angle be trisected? This fact encouraged mathematicians to believe that a general angle trisection method with just a straight edge and compass was within reach.

Hippocrates (460-380 BC), the first person to attempt the trisection of an angle, was unsuccessful but contributed to the world of geometry by labeling points and lines with letters. Hippias (460-399 BC) formed the first successful trisection with a curve called quadratix (this was the first curve introduced in geometry besides lines and circles). Menaechmus (380-320 BC) discovered conic sections. Archimedes (287-217 BC) was able to produce an uncomplicated solution but it required the use of a marked straightedge twice in order to complete the trisection. Nicomedes (280-210 BC) also used a mark straightedge in one of his

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techniques, but succeeded in the trisection using a conchoid. Apolonius (250-175 BC) discovered that by using conic sections, trisection was possible. Both Pappus (early fourth century) and Descartes (1596-1650) used Apolonius' discovery to trisect an angle with a hyperbola and parabola, respectively. However, the problem of trisecting an angle with only an unmarked straightedge and compass remained.

Beginning in the sixteenth century, mathematicians utilized the concepts of the Greek constructions used in trisection and doubling the volume of the cube to solve cubic and quartic equations. Francios Viete (1540-1603) noticed a link between the equations and trigonometry when working with irreducible polynomials, which were at the heart of the proof of planar constructible impossibility. Pierre Wantzel (1814-1848) proved the problem of trisection impossible with planar constructions in his 1837 article titled *Research on the Means of Knowing If a Problem of Geometry Can Be Solved with Compass and Straight Edge.* Essentially, Wantzel's proof was based on the fact that an angle trisection was equivalent to being able to construct roots of a cubic equation. This provided a definitive answer to the problem posed 2200 years later. The impossibility of the trisection using planar constructions then gave birth to modern abstract algebra.

2. Basic Constructions

Constructions are divided planar and solid constructions. Planar constructions are done with a compass and straight edge. Solid constructions add conic sections to the set of things solidly constructible (i.e. hyperbolas). In this project, we will not explain how to perform solid constructions. We will simply state, "construct the hyperbola."

2.1. **Notation.**

Before we begin listing constructions needed in trisecting an angle, let us define some notation needed in the following proofs of these constructions:

- $\overleftrightarrow{XY}$ denotes the line through points X and Y.
- $\overline{XY}$ denotes the segment from point X to point Y.
- $C(X,Y)$ denotes a circle with center X and radius $\overline{XY}$.
- XY denotes the magnitude of segment $\overline{XY}$
- $\angle XYZ$ denotes the measure or name of an angle, depending on the context.

Let us start with the following axioms of planar constructions:

(1) Given two points A and B, we can construct the line and the segment that passes through points A and B (i.e. $\overleftrightarrow{AB}$ and $\overline{AB}$ respectively),

(2) Given two points A and B, we can construct the circle with center A and radius $\overline{AB}$. Denote the circle by $C(A, B)$,
(3) Given a line through the points A and B and an angle θ, we can construct angle θ such that one of its rays lies on the line through points A and B.

Now we can proceed to proving propositions needed to produce the trisection of an angle using a hyperbolic curve,

Proposition 2.1. *(**Rusty Compass Theorem**) Given points A, B, and C, we wish to construct a circle centered at point A with radius equal to BC* [8].

Proof. First, draw $C(A, B)$ and $C(B, A)$ and obtain point D which forms equilateral $\triangle ABD$. Then, construct $C(B, C)$. Extend $\overline{DB}$ past point B and call $\overline{DB} \cap C(B, C)$, point E. Construct $C(D, E)$. Then extend $\overline{DA}$ past point A and label $\overline{DA} \cap C(D, E)$, point F. Construct $C(A, F)$. Because E lies on $C(B, C)$, $BE = BC$. Then, because $\triangle ABD$ is equilateral, $DA = DB$. Also, because E and F lie on a circle with center D, $DE = DF$. Therefore, $AF = BE = BC$ [2]. $\square$

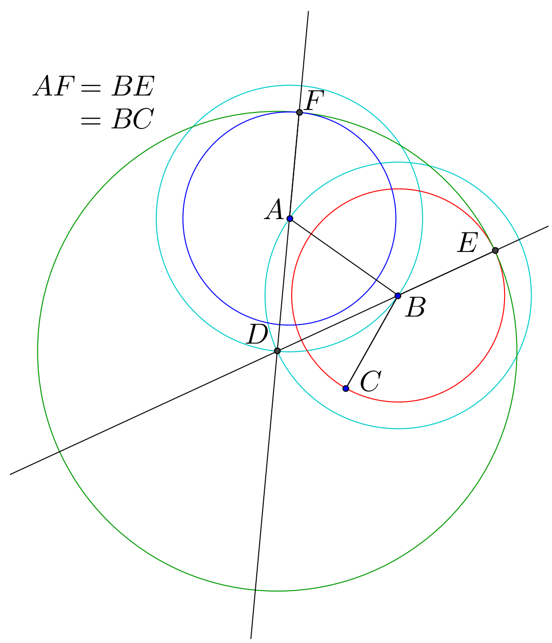

FIGURE 1. Rusty Compass Theorem

Proposition 2.2. *(**Copying an Angle**) Given $\angle ABC$ and a line l containing a point D, we can find E on l and a point F such that $\angle ABC = \angle EDF$* [8].

Proof. Using Proposition 2.1 we construct the point E on line l with $BA = DE$. Construct a circle with center D with radius length BC. Construct another circle with center E with radius length AC. We get the point F. Therefore, $\triangle ABC \cong \triangle EDF$ by SSS. So, $\angle ABC = \angle EDF$. $\square$

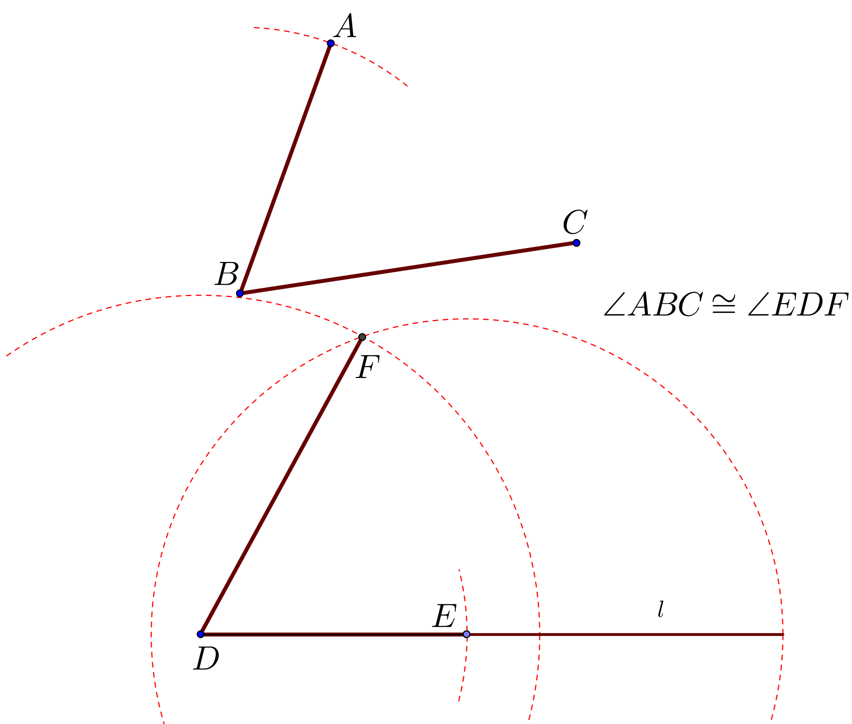

FIGURE 2. Copying an Angle

Proposition 2.3. *(**Bisecting an Angle**) Given $\angle ABC$, there is a point D such that $\angle ABD \cong \angle DBC$* [8].

Proof. Extend $\overline{AB}$ to get the line l. Construct $C(B, C)$ to get the point E on l. Construct $C(E, B)$ and $C(C, B)$ to get D. Then $\angle ABD \cong \angle DBC$. Therefore, $\overline{BD}$ bisects $\angle ABC$. $\square$

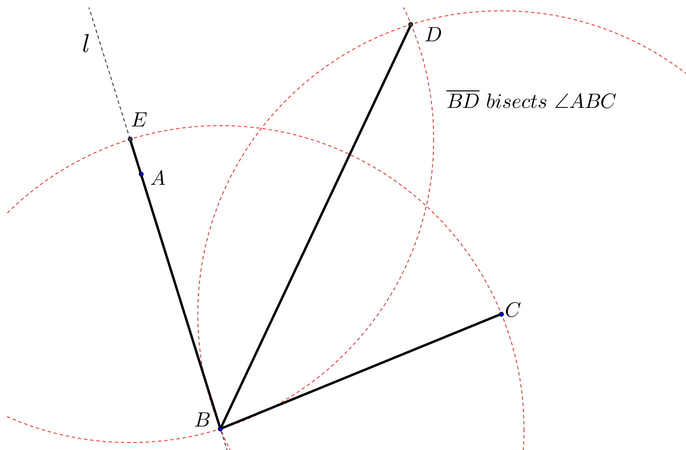

FIGURE 3. Bisecting an Angle

Proposition 2.4. ***(Dropping a Perpendicular)*** *Given a line l and a point p not on l, we can construct a line l' which is perpendicular to l and passes through p* [8].

Proof. There exists a point A on l. If $\overleftrightarrow{pA} \perp l$, we are done. If not, construct $C(p, A)$ to get B. Next, construct $C(A, B)$ and $C(B, A)$ to get C. Then $\overleftrightarrow{pC}$ is perpendicular to l. $\square$

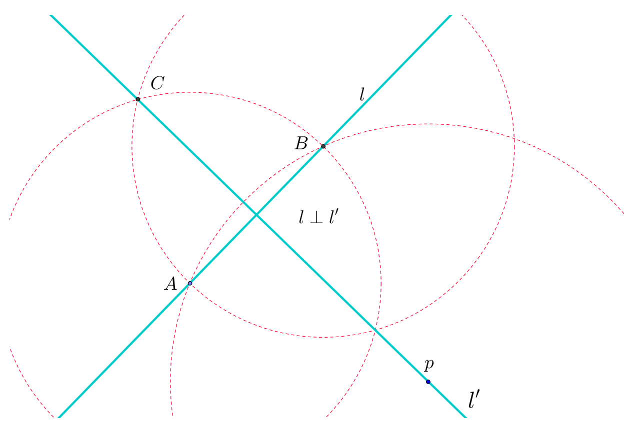

FIGURE 4. Dropping a Perpendicular

Proposition 2.5. ***(Parallel Postulate, Playfair)*** *Given a line l and P not on l, we can construct l' through P and parallel to l* [8].

Proof. Construct l'' such that it is perpendicular to l and passes through P by Proposition 4. Now construct l' through P and perpendicular to l'' by Proposition 6. l' is parallel to l through P. $\square$

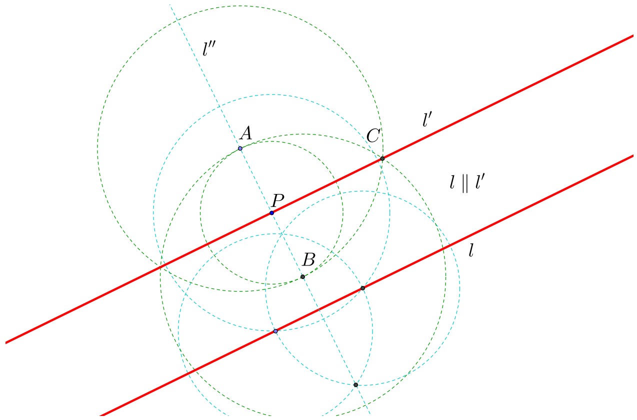

FIGURE 5. Parallel Postulate, Playfair

Proposition 2.6. ***(Raising a Perpendicular)*** *Given a point p on a line l, then you can construct l' through point p perpendicular to line l [8].*

Proof. There exists a point A on line l distinct from point p. Construct $C(p, A)$ to obtain $l \cap C(p, A) = B$. Construct $C(A, B)$ and $C(B, A)$ to get C. Draw $\overleftrightarrow{pC}$. This line is l'. $\square$

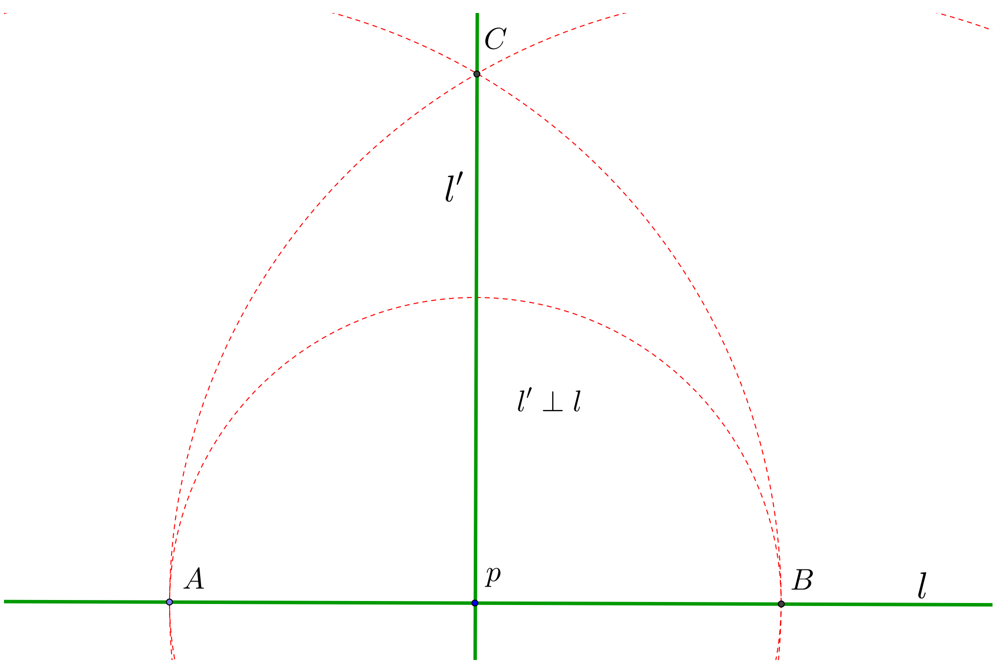

FIGURE 6. Raising a Perpendicular

3. The Hyperbola

3.1. **Defintion.** In general, conic sections are curves (except in the cases of degenerate sections) that are obtained by intersecting a plane with a cone. These sections include circles, ellipses, hyperbolas, and parabolas [7].

Hyperbolas are typically defined in two ways:

- The *locus* of all points P in the plane the difference of whose distances $r_1 = \overline{F_1P}$ and $r_2 = \overline{F_2P}$ from two fixed points, called *foci*, is a constant k, with $k = r_2 - r_1$ (as seen in Figure 7) [4],
- The *locus* of all points for which the ratio of distances from one *focus* to a line (*directrix*) is a constant e (*eccentricity*), with $e > 1$ (as seen in Figure 8) [3].

Hyperbolas, specifically, are created by intersecting a plane with a double cone (with apexes touching) generated by a line with an angle θ from the x-axis. Slice this double cone with a constant plane parallel to the axis of the double cone at an angle ϕ such that $\theta < \phi < \frac{\pi}{2}$ to obtain a hyperbola [3]. These *loci* create two distinct branches of the curve.

3.2. **General Definition of Features.** Hyperbolas have two axes: the *transverse axis* and the *conjugate axis*. To locate the *transverse axis*, the two *vertices* must be considered. The *vertices* can be described as the points where the distance between the two branches is the least, one *vertex* on each branch of the hyperbola. The *transverse axis* is the line segment which connects the *vertices*. The *conjugate axis* is perpendicular to the *transverse axis* and runs through the *center* of the hyperbola which is located at the midpoint of the *transverse axis* and the vertices (usually denoted by (h, k)). It is also important to note that the *center*, (h, k), of a hyperbola is located at the intersection of the *asymptotes*, *conjugate axis*, and *transverse axis*.

The hyperbola is an open curve, which means that it extends to infinity. Zooming out on the curve, you will see the curve approaching a pair of lines called the *asymptotes*.

The symmetry about its *conjugate* and *transverse axis* is an interesting characteristic of the hyperbola.

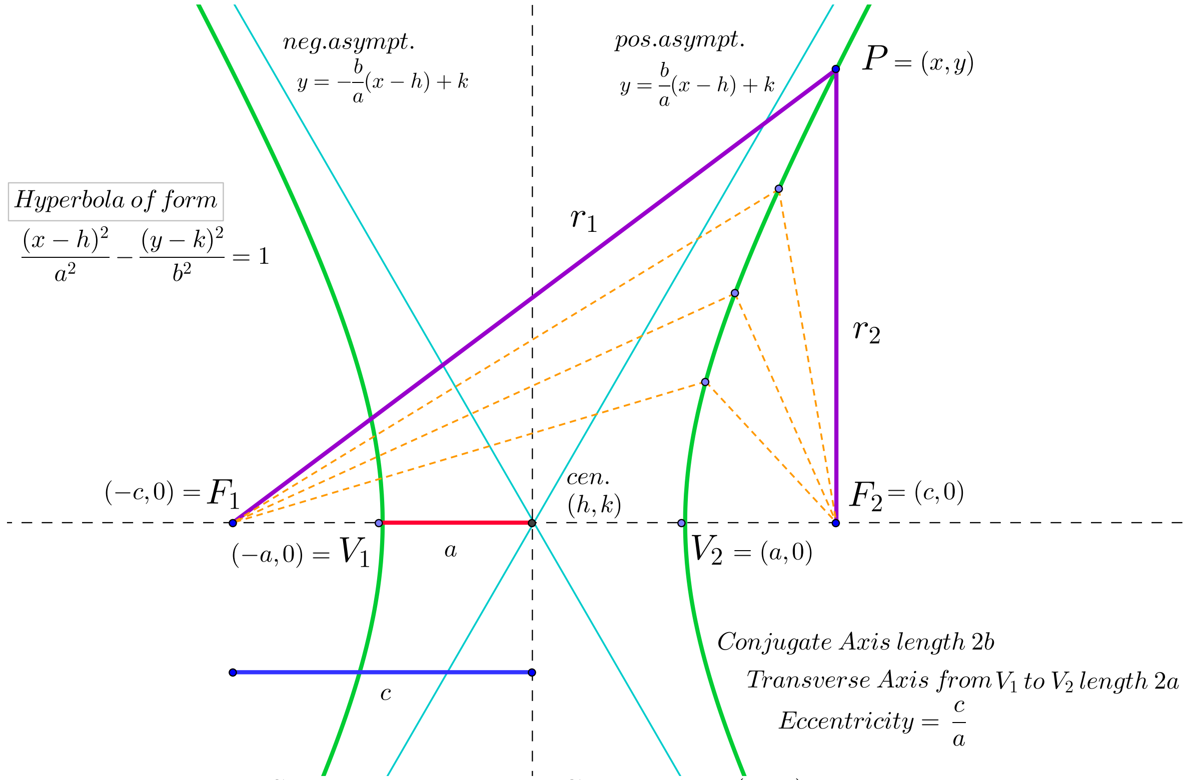

FIGURE 7. Standard Hyperbola Centered at (h, k) and its Features

3.2.1. *Equations of Features of a Hyperbola.* Hyperbolas are algebraically defined by an equation of the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

. with real coefficients and solutions in the form of ordered pairs, (x, y). Also, hyperbolas are commonly expressed in standard form.

Two standard forms of hyperbolas are [6]:

- If the transverse axis is along the x - axis in the Cartesian plane (as in Figure 7) and centered at (h, k), the equation can be written as
$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1.$$
- If the transverse axis is along the y - axis in the Cartesian plane and centered at (h, k), the equation can be written as
$$\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1.$$

By observing the standard forms of a hyperbola, you can see that the a^2 term always corresponds to the x^2 term and b^2 always to the y^2 term. Thus, we can determine a and b directly from the standard form of the hyperbola. Also, c can be determined by the following equation $c = \sqrt{a^2 + b^2}$.

The values a and c define where the foci and vertices lie on the hyperbola's horizontal axis. As seen in Figure 7, the foci lie at distances a from the center, and the vertices lie at distances c from the center. Hence, the coordinates of the vertices and foci are $(a, 0)$, $(-a, 0)$ and $(c, 0)$, $(-c, 0)$ respectively.

As seen in Figure 7, the asymptotes of hyperbolas centered at (h, k) with different transverse axes are given by the following equations.

- For hyperbolas with transverse axis along the x-axis,

$$y = \pm\frac{b}{a}(x - h) + k.$$

- For hyperbolas with transverse axis along the y-axis,

$$y = \pm\frac{a}{b}(x - h) + k.$$

The conjugate and transverse axes can also be expressed as functions of a and b. The equations are $2b$ and $2a$ respectively.

The shape of the hyperbola is characterized by e (eccentricity), which can be defined as the ratio of the distances from a focus to the corresponding directrix (as seen in Figure 8). The eccentricity is given by the equation $e = \frac{c}{a}$ and equations for the directrix are given by $x = \frac{a^2}{c}$ and $x = \frac{b^2}{c}$.

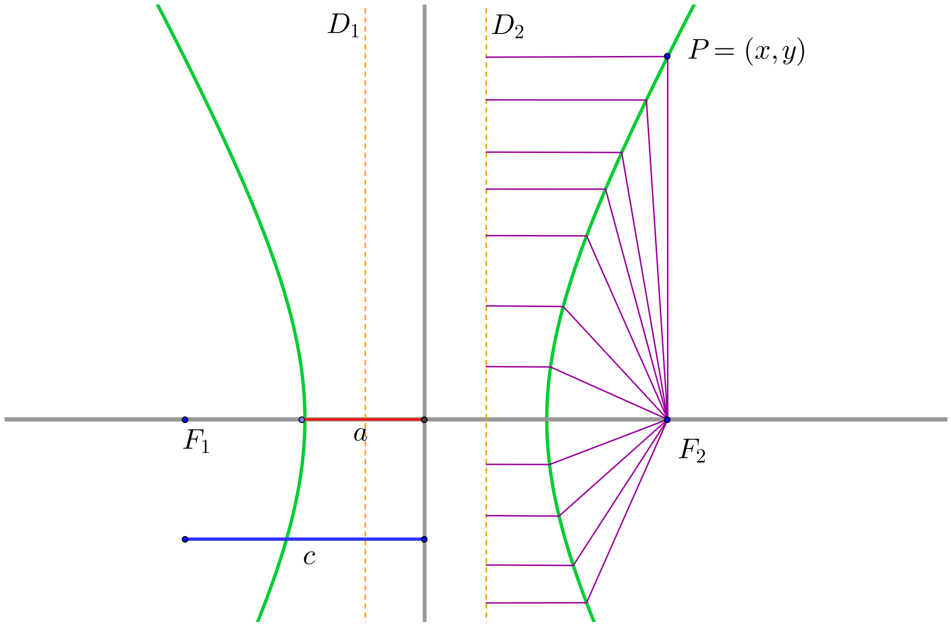

FIGURE 8. Hyperbola Showing Eccentricity Ratio

Now that we have defined and given equations for all features of a hyperbola, we can begin to derive the specific hyperbola needed for the purpose of this project.

3.3. **Derivation of General Hyperbola.** We will now show the derivation of the curve that consists of all the points $C = (x, y)$, in $\triangle ABC$, such that $\angle CBA = 2\angle CAB$.

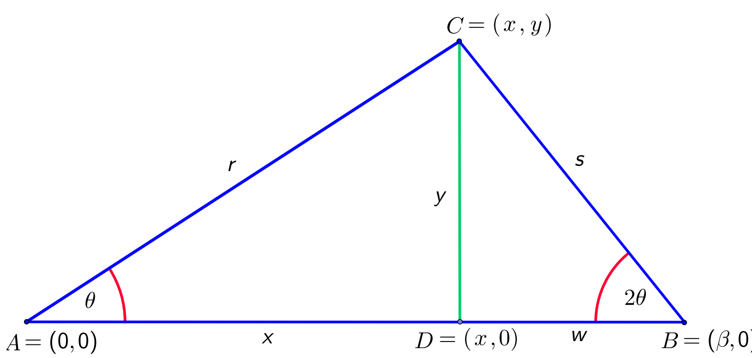

FIGURE 9. Triangle Used for Equation Derivation

From Figure 9, we can see the following,

$$\beta = x + w \tag{3.1}$$
$$y = r\sin\theta \tag{3.2}$$
$$x = r\cos\theta \tag{3.3}$$
$$s = \frac{r\sin\theta}{\sin 2\theta} \tag{3.4}$$

Using the Pythagorean Theorem to find side w,

$$y^2 + w^2 = s^2$$
$$w^2 = s^2 - y^2$$

Substituting y and s values from equations (3.2) and (3.4),

$$\begin{aligned} w^2 &= \left(\frac{r\sin\theta}{\sin 2\theta}\right)^2 - (r\sin\theta)^2 \\ &= \frac{r^2\sin^2\theta}{\sin^2 2\theta} - r^2\sin^2\theta \end{aligned}$$

Taking the square root of both sides of our equation,

$$\begin{aligned} w &= \sqrt{\frac{r^2\sin^2\theta}{\sin^2 2\theta} - r^2\sin^2\theta} \\ &= \sqrt{r^2\sin^2\theta\left(\frac{1}{\sin^2 2\theta} - 1\right)} \\ &= r\sin\theta\sqrt{\left(\frac{1}{\sin^2 2\theta} - 1\right)} \\ &= r\sin\theta\sqrt{\csc^2 2\theta - 1} \\ &= r\sin\theta\sqrt{\cot^2 2\theta} \\ &= r\sin\theta\cot 2\theta \\ &= r\sin\theta\left(\frac{\cos 2\theta}{\sin 2\theta}\right) \end{aligned}$$

$$\begin{aligned}
&= r\sin\theta\Big(\frac{\cos 2\theta}{2\sin\theta\cos\theta}\Big)\\
&= \frac{r\cos 2\theta}{2\cos\theta}\\
&= \frac{r(\cos^2\theta - \sin^2\theta)}{2\cos\theta}\\
&= \frac{r\cos^2\theta}{2\cos\theta} - \frac{r\sin^2\theta}{2\cos\theta}
\end{aligned}$$

$$w = \frac{r\cos\theta}{2} - \frac{r\sin^2\theta}{2\cos\theta} \tag{3.5}$$

Taking our expression in equation (3.3) and (3.5) into equation (3.1),

$$\begin{aligned}
\beta &= x + w\\
&= r\cos\theta + \frac{r\cos\theta}{2} - \frac{r\sin^2\theta}{2\cos\theta}
\end{aligned}$$

Multiplying both sides of the equation by $2r\cos\theta$,

$$2r^2\cos^2\theta + r^2\cos^2\theta - r^2\sin^2 = 2\beta r\cos\theta.$$

Using substitution with equations (3.2) and (3.3),

$$2x^2 + x^2 - y^2 = 2\beta x.$$

Completing the square,

$$\begin{aligned}
3x^2 - 2\beta x - y^2 &= 0\\
x^2 - \frac{2}{3}\beta x - \frac{1}{3}y^2 &= 0\\
(x - \frac{\beta}{3})^2 - \frac{y^2}{3} &= \frac{\beta^2}{9}
\end{aligned}$$

Putting the formula for this hyperbola in standard form, we arrive at the curve,

$$\frac{(x - \frac{1}{3}\beta)^2}{\left(\frac{\beta}{3}\right)^2} - \frac{y^2}{\left(\frac{\beta}{\sqrt{3}}\right)^2} = 1. \tag{3.6}$$

This equation satisfies the conditions needed for the problem's purpose. This hyperbola is missing a point because when the point C on the $\triangle ABC$ is collapsed towards $\overline{AB}$, it will eventually become a point on $\overline{AB}$. Then, it can no longer satisfy the condition $\angle CBA = 2\angle CAB$.

3.4. **Hyperbola with $\beta = 1$.** For this project, our desired curve is Γ. It is the locus of all points $C = (x, y)$ such that, in $\triangle ABC$ with $A = (0, 0)$ and $B = (1, 0)$, $\angle CBA = 2\angle CAB$ (as seen in Figure 9). We will obtain Γ by substituting $\beta = 1$ into equation (3.6). We arrive at the equation,

$$\frac{(x-\frac{1}{3})^2}{\left(\frac{1}{3}\right)^2} - \frac{y^2}{\left(\frac{1}{\sqrt{3}}\right)^2} = 1 \tag{3.7}$$

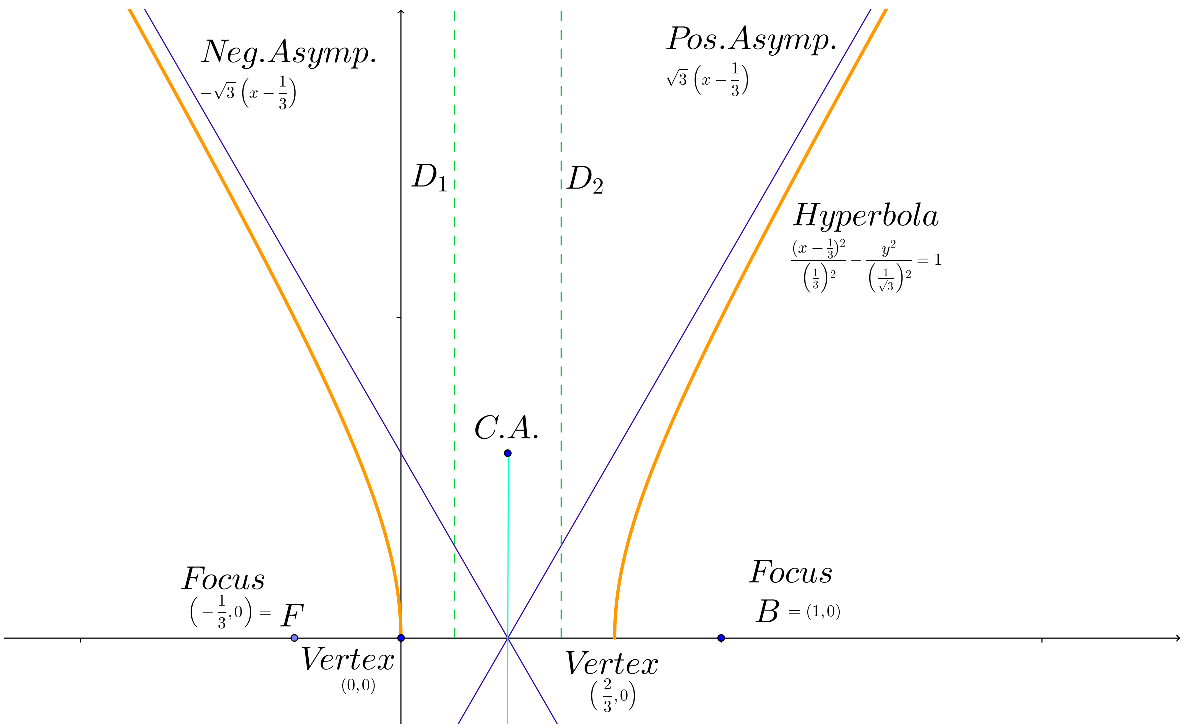

FIGURE 10. Depiction of Γ and its Features

3.4.1. *Equations of Features of* Γ. Equation (3.7) is in standard form because it is of the form,

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1.$$

Most importantly, we see that Γ is centered at $(\frac{1}{3}, 0)$ instead of the origin (as seen in Figure 10) . Because of this, we must shift the vertices and foci right $\frac{1}{3}$ units. Therefore, the vertices have coordinates $(-a+\frac{1}{3}, 0)$ and $(a+\frac{1}{3}, 0)$. Shifting the equations of the foci in the same fashion, we arrive at the coordinates $(-c + \frac{1}{3}, 0)$ and $(c + \frac{1}{3}, 0)$. We obtain vertices: $(0, 0)$ and $(\frac{2}{3}, 0)$, and foci: $(\frac{-1}{3}, 0)$ and $(1, 0)$. For the problem of the trisection, we will only use the right branch of Γ along with the right focus and vertex.

From the standard form of Γ, we observe that its center is located at $(\frac{1}{3}, 0)$ with $a = \frac{1}{3}$ and $b = \frac{1}{\sqrt{3}}$.

Using the relationship $c = \sqrt{a^2 + b^2}$ to find c, we obtain $c = \frac{2}{3}$. The values of a and c are used in finding the eccentricity of the Γ,

$$e = \frac{c}{a} = \frac{\frac{2}{3}}{\frac{1}{3}} = 2.$$

It also has transverse axis along the x-axis in the Cartesian plane with length $\frac{2}{3}$ and conjugate axis parallel to the y-axis in the Cartesian plane with length $\frac{2}{\sqrt{3}}$. Lastly, the asymptotes of Γ are denoted by equations, $y = \sqrt{3}(x - \frac{1}{3})$ and $y = -\sqrt{3}(x - \frac{1}{3})$. Now that we have quantified the features of Γ, we can proceed to our problem: trisecting an arbitrary acute angle using Γ.

4. Trisecting the Angle

Our argument is based on an argument presented by Pappus in Sir Thomas Heath's A History of Greek Mathematics [1].

4.1. **Construction of the Trisection.**

Given $\overline{AB}$ (with $A = (0, 0)$ and $B = (1, 0)$ on the Cartesian plane) and an angle θ, we can construct the trisection of an arbitrary angle using our derived hyperbola, Γ, along with basic constructions outlined in Section 2 [1].

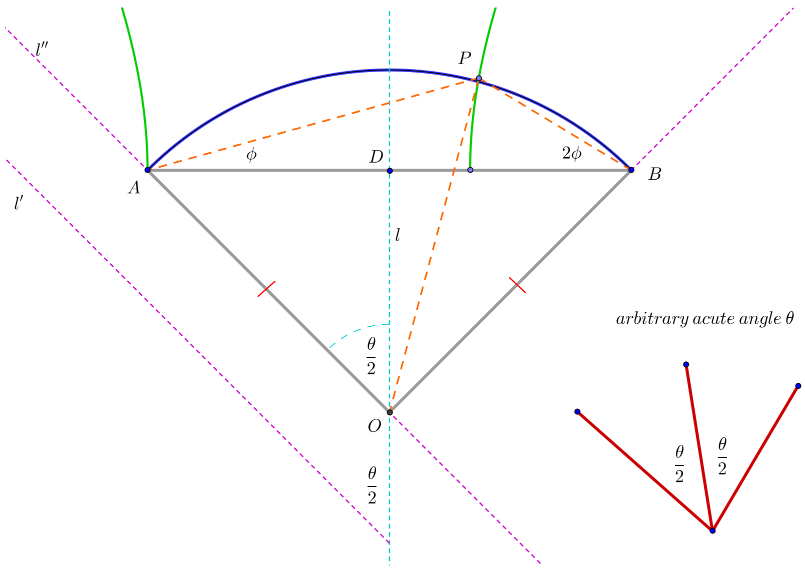

FIGURE 11. Trisection Construction

(1) Construct the hyperbola Γ with right branch's focus at point B.
(2) Use Proposition 2.1 to construct point D at $(\frac{1}{2}, 0)$.
(3) Using Proposition 4, construct l perpendicular to $\overline{AB}$ at point D.
   - Note that l is the perpendicular bisector of $\overline{AB}$.
(4) Bisect angle θ using Proposition 2.3 to obtain $\frac{\theta}{2}$.
(5) From the axioms of basic planar constructions, we are able to construct ray l' with right endpoint on l at an angle $\frac{\theta}{2}$ from l measured anti-clockwise.
   - If l' contains point A, continue from **Step (7)**.
   - If l' does not contain A, continue from **Step (6)**.
(6) Construct $l'' \parallel l'$ through point A.
(7) Obtain point O such that $\angle AOD = \frac{\theta}{2}$.
(8) Construct $\overline{AO}$.
(9) Using Proposition 2.2, reflect $\angle AOD$ about line l such that it creates $\angle DOB$.
(10) Construct $\overline{OB}$.
   - Note that $AO = OB$, therefore $\triangle AOB$ is isosceles.
(11) Construct $C(O, A)$.
   - Note that $C(O, A)$ contains both points A and B because $\overline{OA}$ and $\overline{OB}$ are radii. Also, $\angle AOB = \theta$.
(12) Obtain point P from the intersection of $\overset{\frown}{AB}$ and Γ.

(13) Construct $\overline{OP}$, $\overline{AP}$, and $\overline{PB}$.
- Note that $\triangle APB$, obtained in previous step, is the triangle such that $2\angle PAB = \angle PBA$.

4.2. **Proof of Trisection.** For the sake of the ease of the proof we will rename the central angles α and ψ and note that $\alpha + \psi = \theta$. Let $\angle PAB = \phi$, $\angle PBA = 2\phi$, $\angle AOP = \alpha$, and $\angle POB = \psi$.

Theorem 4.1. *If $\angle PBA = 2\angle PAB$, then $\angle AOP = 2\angle POB$. Hence, $\angle POB$ trisects $\angle AOB$.*

Proof. We know that $\frac{1}{2}\psi = \phi$ and $2\phi = \frac{1}{2}\alpha$ [8]. Solving for ϕ in both equations and equating them, we arrive at $\frac{1}{2}\psi = \frac{1}{4}\alpha$. So, $2\psi = \alpha$ or $\psi = \frac{1}{2}\alpha$. Notice that $\alpha + \psi = \angle AOB$, so $\angle AOB = 3\psi$ or $\psi = \frac{1}{3}\angle AOB$. Therefore, ψ trisects $\angle AOB$. □

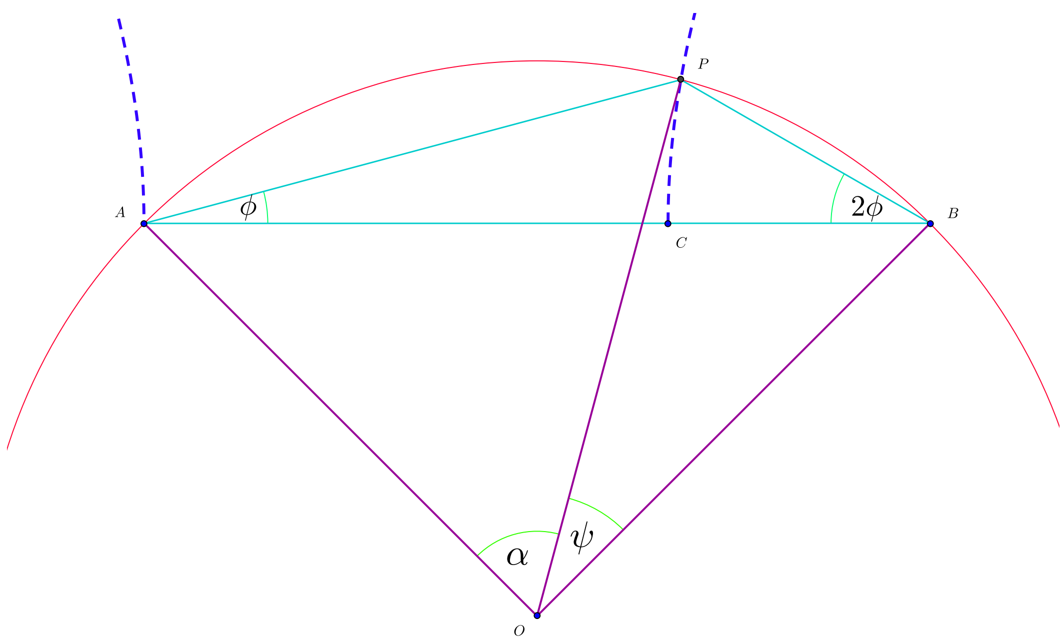

FIGURE 12. Depiction of Trisection Proof

We have successfully constructed an angle given a line segment and trisected it using Γ.

5. GEOMETRIC INTERPRETATION OF FEATURES

We will look at how the asymptotes, right directrix, and right focus of Γ are geometrically interpreted in the picture of the trisection done in Section 4.

First, we will show that the intersection of the asymptotes occurs at $\frac{1}{3}$ the distance from point A on $\overline{AB}$, giving it coordinates $(\frac{1}{3}, 0)$. This is done by setting the equations found in Section 3.4.1 equal to each other.

$$\begin{aligned}
\sqrt{3}\Big(x - \frac{1}{3}\Big) &= -\sqrt{3}\Big(x - \frac{1}{3}\Big) \\
\sqrt{3}x - \frac{\sqrt{3}}{3} &= -\sqrt{3}x + \frac{\sqrt{3}}{3} \\
2\sqrt{3}x &= \frac{2\sqrt{3}}{3} \\
x &= \frac{1}{3}
\end{aligned}$$

Second, we will show that the directrix corresponding to the right focus of Γ lies halfway between A and B (i.e. also note that the directrix of Γ is perpendicular to $\overline{AB}$). By using the equation for the directrix to derive the coordinates,

$$\begin{aligned}
x &= \frac{b^2}{c} \\
x &= \frac{\frac{1}{3}}{\frac{2}{3}} \\
x &= \frac{1}{2}.
\end{aligned}$$

Since the directrix of Γ divides $\overline{AB}$ into two equal parts and is perpendicular to $\overline{AB}$, making it the perpendicular bisector of $\overline{AB}$. Consequently, it (also) bisects $\angle KOP$ which is the middle angle of trisection of $\angle AOB$ as seen in Figure 13.

If you just look at the right branch of Γ (as used in the trisection), the point B is the focus corresponding to the branch.

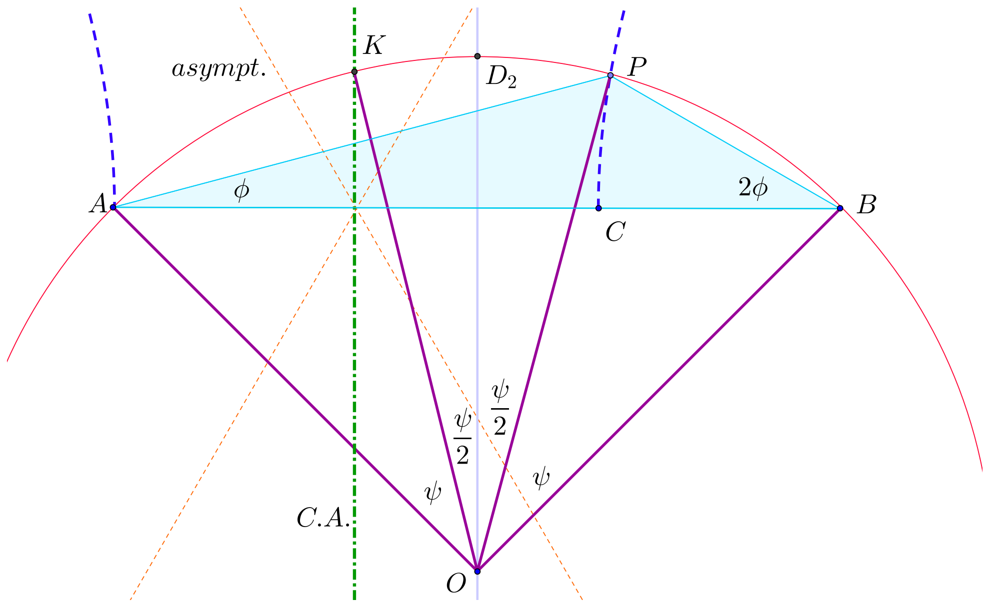

FIGURE 13. Geometric Interpretations of Features of Γ

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Mathematics Department, University of Louisiana at Lafayette, Lafayette, Louisiana
E-mail address: njm5765@louisiana.edu

Mathematics Department, Southern University, Baton Rouge, Louisiana
E-mail address: tiaburden@yahoo.com

Mathematics Department, Tulane University, New Orleans, Louisiana
E-mail address: jfink@tulane.edu