Curves and Sectioning Angles

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- From Greek constructions to abstract algebra

Notation

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- $C(X, Y)$ denotes a circle with center X and radius $\overline{XY}$.
- XY denotes the magnitude of segment $\overline{XY}$
- $\angle XYZ$ denotes the measure or name of an angle, depending on the context.

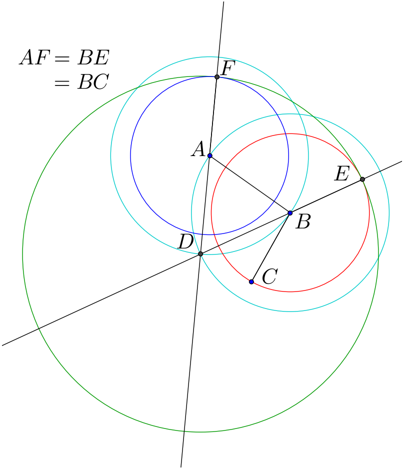
Propositions

Here are the propositions used in our final trisection construction along with pictures demonstrating construction procedures:

- Rusty Compass Theorem
- Copying an Angle
- Bisecting an Angle
- Parallel Postulate
- Raising a Perpendicular

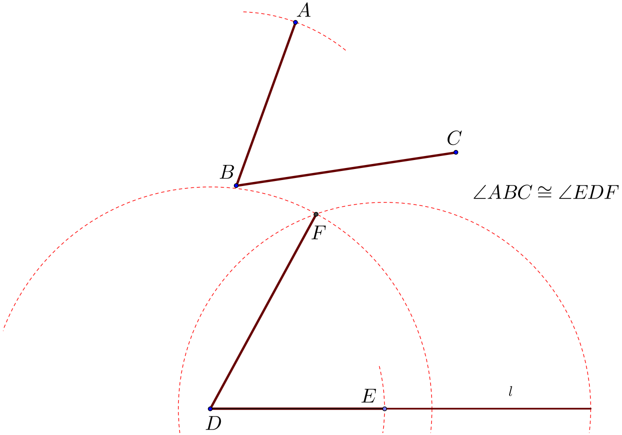
Rusty Compass Theorem

Given points A, B, and C, we wish to construct a circle centered at point A with radius equal to BC.


$AF = BE$
$= BC$
F
A
E
B
D
C

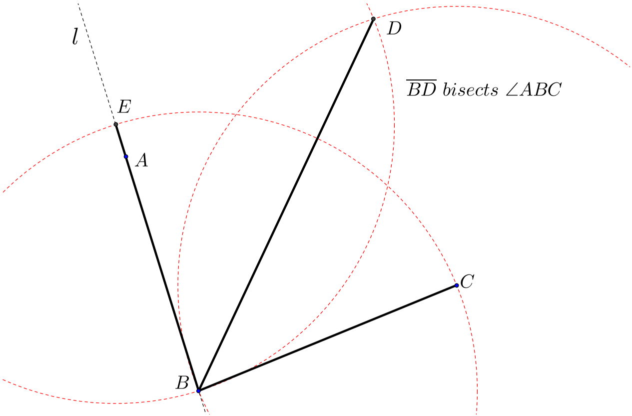
Copying an Angle

Given $\angle ABC$ and a line l containing a point D, we can find E on l and a point F such that $\angle ABC = \angle EDF$.


A
C
B
$\angle ABC \cong \angle EDF$
F
E
l
D

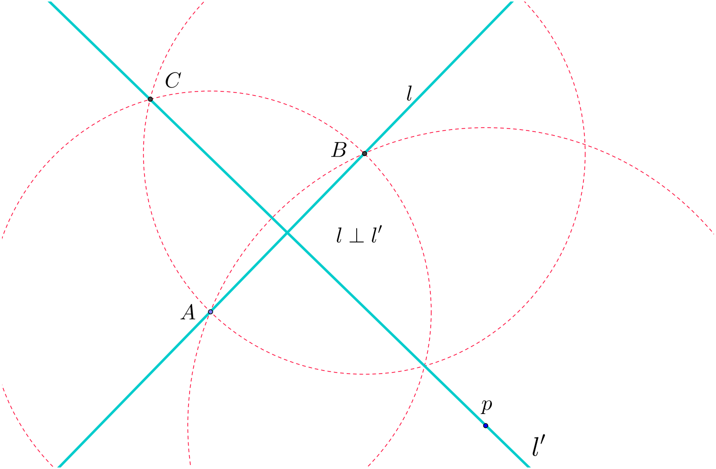
Bisecting an Angle

Given $\angle ABC$, there is a point D such that $\angle ABD \cong \angle DBC$.


l
D
$\overline{BD}$ *bisects* $\angle ABC$
E
A
C
B

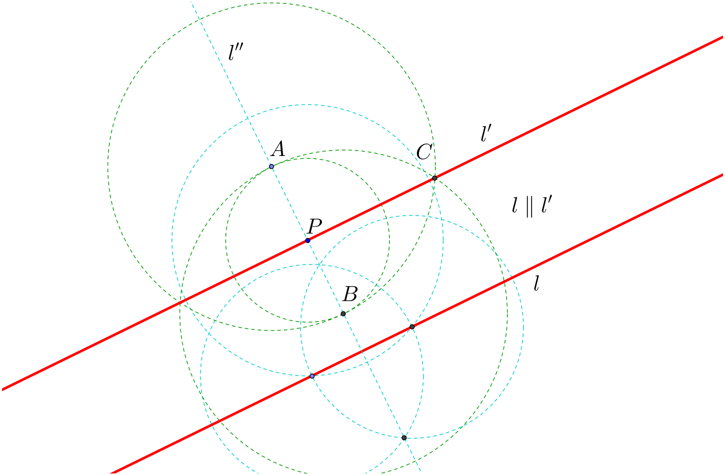
Dropping a Perpendicular

Given a line l and a point p not on l, we can construct a line l' which is perpendicular to l and passes through p.


C
l
B
$l \perp l'$
A
p
l'

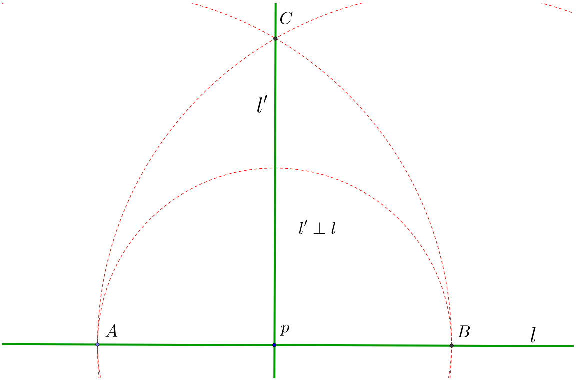
Parallel Postulate (Playfair)

Given a line l and P not on l, we can construct l' through P and parallel to l.


l''
A
C
l'
$l \parallel l'$
P
B
l

Raising a Perpendicular

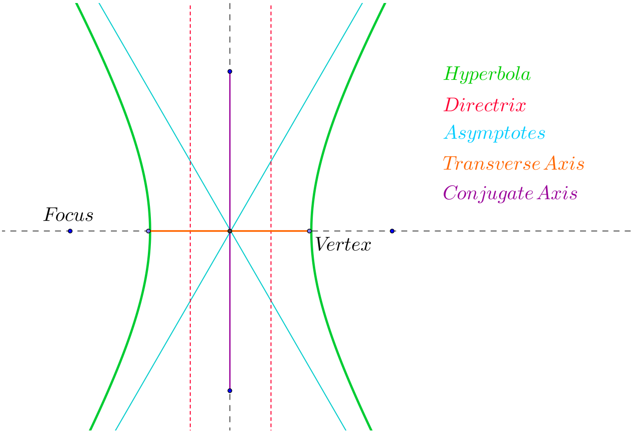
Given a point p on a line l, you can construct l' through point p perpendicular to line l.


C
l'
$l' \perp l$
A
p
B
l

Important Features of Hyperbolas

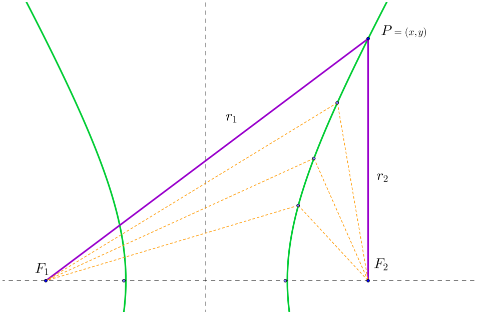
Important Features are:

1. *focus*
2. *directrix*
3. *vertex*
4. *asymptotes*
5. *transverse axis*
6. *conjugate axis*



Definition of Hyperbolas

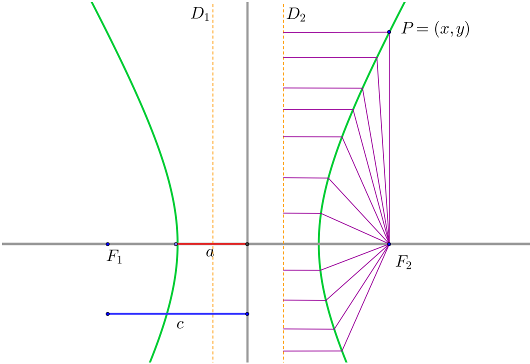
The *locus* of all points P in the plane the difference of whose distances $r_1 = \overline{F_1P}$ and $r_2 = \overline{F_2P}$ from two fixed points, called *foci*, is a constant k, with $k = r_2 - r_1$.



Definition of Hyperbolas

The *locus* of all points for which the ratio of distances from one *focus* to a line (the *directrix*) is a constant e (the *eccentricity*), with $e > 1$.

- These loci create two distinct branches of the curve.



From Definitions to Derivation

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- Derive a curve that satisfies the specific conditions of the problem.

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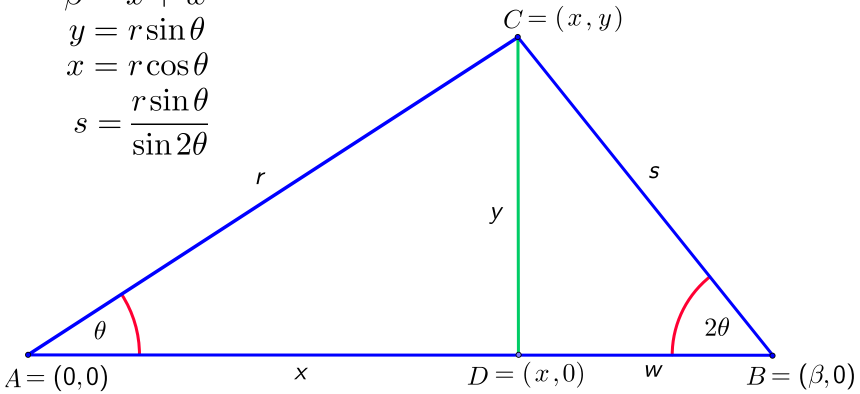
So how do we tie all of this together?

- Derive a curve that satisfies the specific conditions of the problem.
- Use this curve to trisect an arbitrary acute angle.
- Interpret the features of the curve as it fits into the trisection picture.

Process of Deriving Γ

$$\beta = x + w$$
$$y = r\sin\theta$$
$$x = r\cos\theta$$
$$s = \frac{r\sin\theta}{\sin 2\theta}$$



Solving for w

Values from Derivation Triangle

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$$w^2 = s^2 - y^2$$

$$w^2 = \frac{r^2\sin^2\theta}{\sin^2 2\theta} - r^2\sin^2\theta$$

$$w = \sqrt{r^2\sin^2\theta\left(\frac{1}{\sin^2 2\theta} - 1\right)}$$

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$$= r\sin\theta\sqrt{\csc^2 2\theta - 1}$$

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$$= r\sin\theta\sqrt{\csc^2 2\theta - 1}$$

$$= r\sin\theta\sqrt{\cot^2 2\theta}$$

$$= r\sin\theta\left(\frac{\cos 2\theta}{2\sin\theta\cos\theta}\right)$$

$$= \frac{r(\cos^2\theta - \sin^2\theta)}{2\cos\theta}$$

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$$w^2 = \frac{r^2\sin^2\theta}{\sin^2 2\theta} - r^2\sin^2\theta$$

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$$= r\sin\theta\sqrt{\cot^2 2\theta}$$

$$= r\sin\theta\left(\frac{\cos 2\theta}{2\sin\theta\cos\theta}\right)$$

$$= \frac{r(\cos^2\theta - \sin^2\theta)}{2\cos\theta}$$

$$w = \frac{r\cos\theta}{2} - \frac{r\sin^2\theta}{2\cos\theta}$$

Solving for β

$$\beta = r\cos\theta + \frac{r\cos\theta}{2} - \frac{r\sin^2\theta}{2\cos\theta}$$

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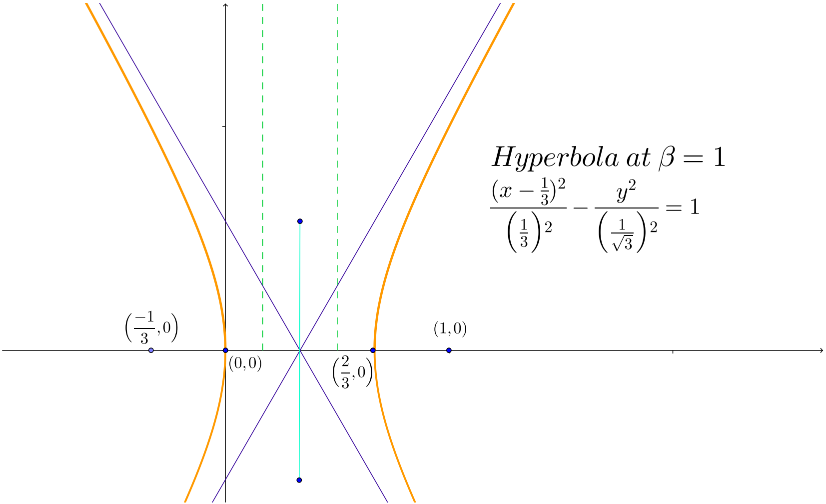
$$\frac{(x-\frac{\beta}{3})^2}{\left(\frac{\beta}{3}\right)^2} - \frac{y^2}{\left(\frac{\beta}{\sqrt{3}}\right)^2} = 1$$

Picture of Γ



Hyperbola at $\beta = 1$

$$\frac{(x-\frac{1}{3})^2}{\left(\frac{1}{3}\right)^2} - \frac{y^2}{\left(\frac{1}{\sqrt{3}}\right)^2} = 1$$

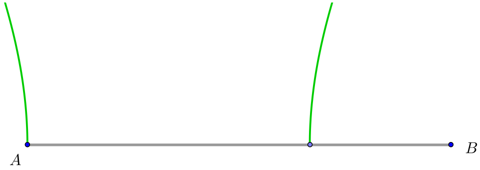
Construction of the Trisection

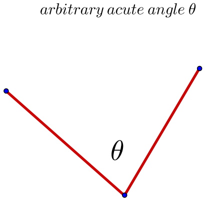
Trisection Construction

Given $\overline{AB}$ (with $A = (0,0)$ and $B = (1,0)$ on the Cartesian plane) and an angle θ, we can construct the trisection of an arbitrary angle using our derived hyperbola, Γ, along with basic constructions previously outlined.

Step 1

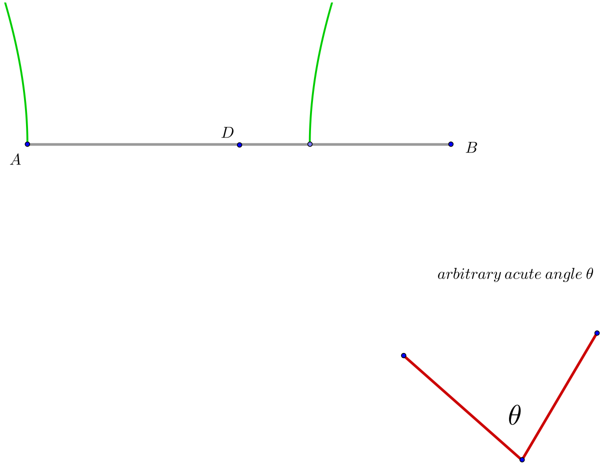
Construct the hyperbola Γ with right branch's focus at point B.





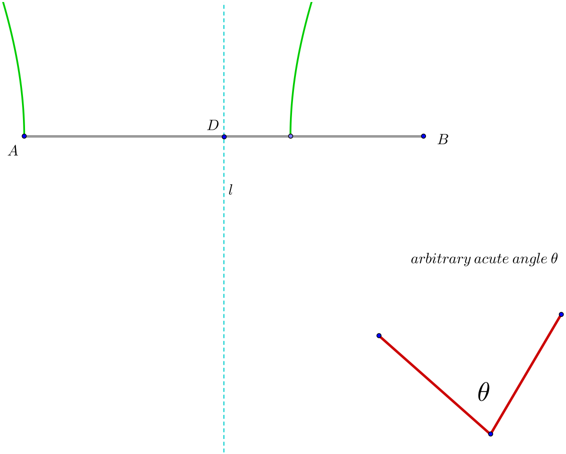
Step 2

Using the Rusty Compass Theorem, construct point D at $(\frac{1}{2}, 0)$.



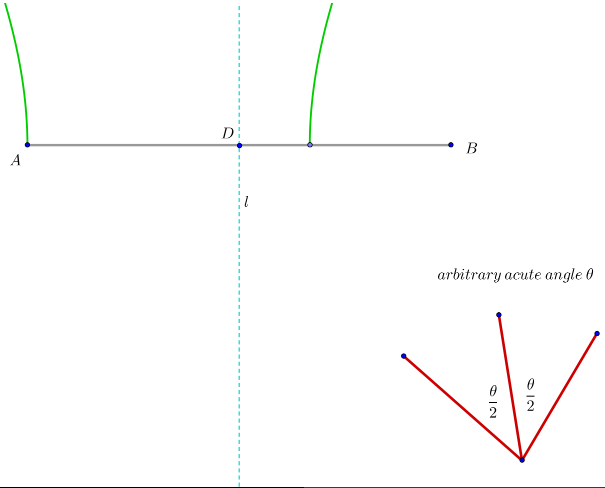
Step 3

Construct line l perpendicular to $\overline{AB}$ at point D by dropping a perpendicular. Note that line l is the perpendicular bisector of $\overline{AB}$.



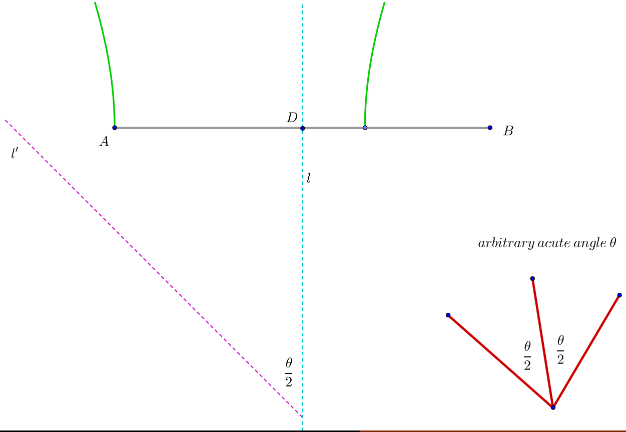
Step 4

Bisect angle θ to obtain $\frac{\theta}{2}$.



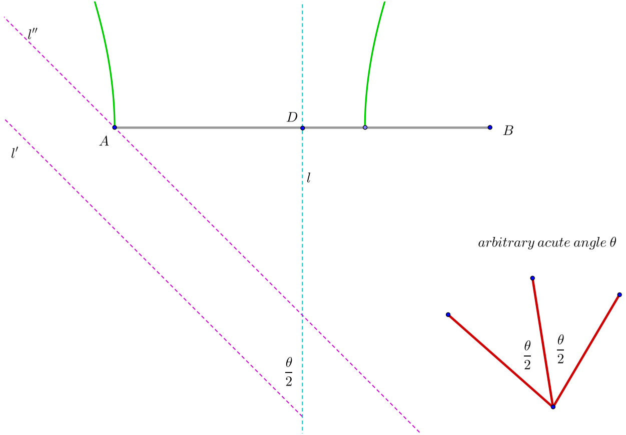
Step 5

From the axioms of basic planar constructions, we are able to construct ray l' with right endpoint on l at an angle $\frac{\theta}{2}$ from l measured anti-clockwise.



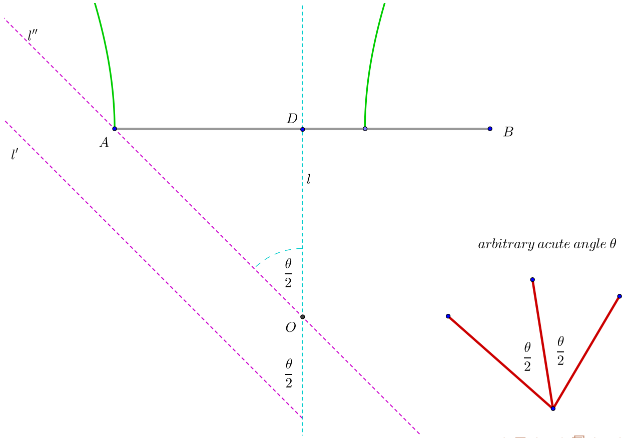
Step 6

If line l' does not contain point A, construct $l'' \parallel l'$ through point A.



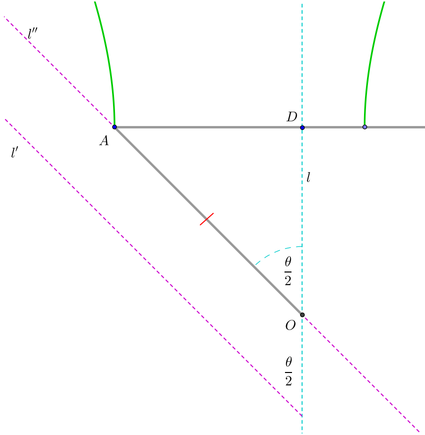
Step 7

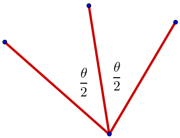
If line l' does contain point A, obtain point O such that $\angle AOD = \frac{\theta}{2}$.



Step 8

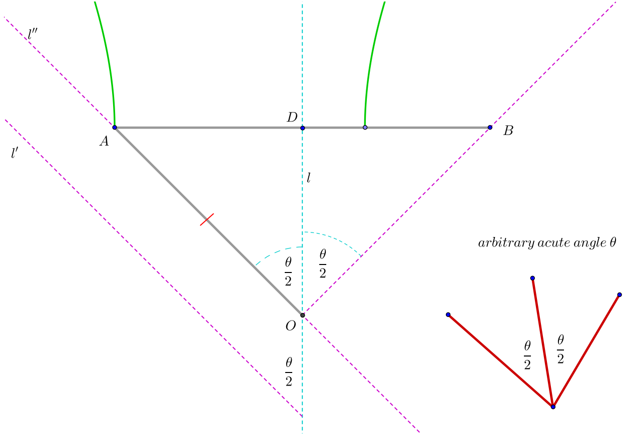
Construct $\overline{AO}$.





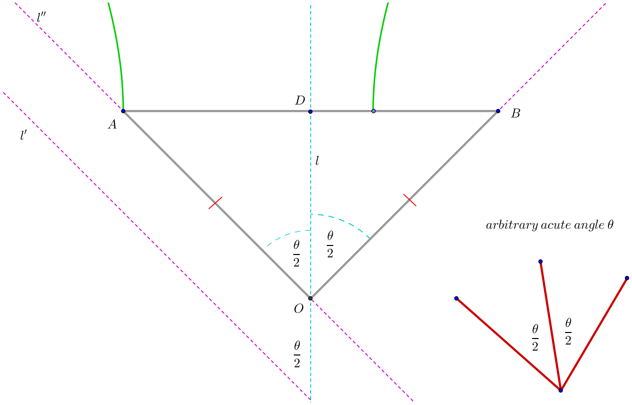
Step 9

Reflect $\angle AOD$ about line l such that it creates $\angle DOB$ by using the construction to copy an angle.



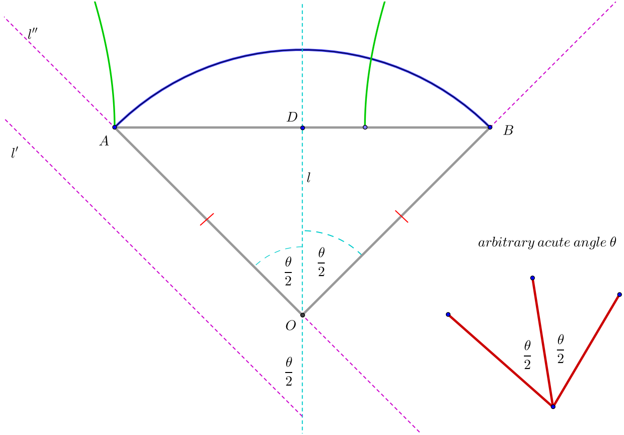
Step 10

Construct $\overline{OB}$. Note that $AO = OB$, so $\triangle AOB$ is isosceles.



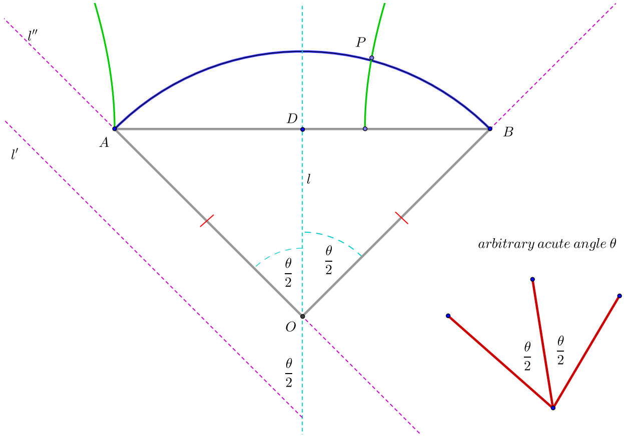
Step 11

Construct $C(O, A)$. Note that $C(O, A)$ contains both points A and B because $\overline{OA}$ and $\overline{OB}$ are radii. Also, $\angle AOB = \theta$.



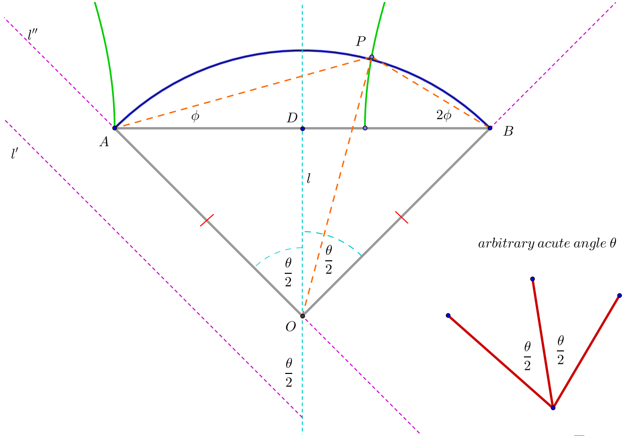
Step 12

Obtain point P from the intersection of $\overset{\frown}{AB}$ and Γ.



Step 13

Construct $\overline{OP}$, $\overline{AP}$, and $\overline{PB}$. Note that $\triangle APB$, obtained in previous step, is the triangle such that $2\angle PAB = \angle PBA$.

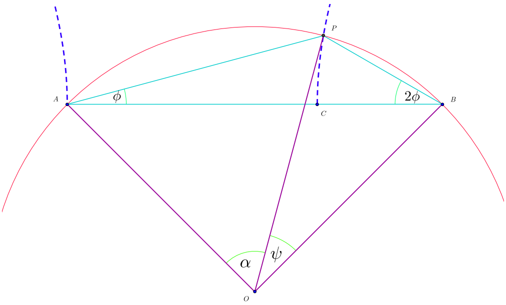


Theorem

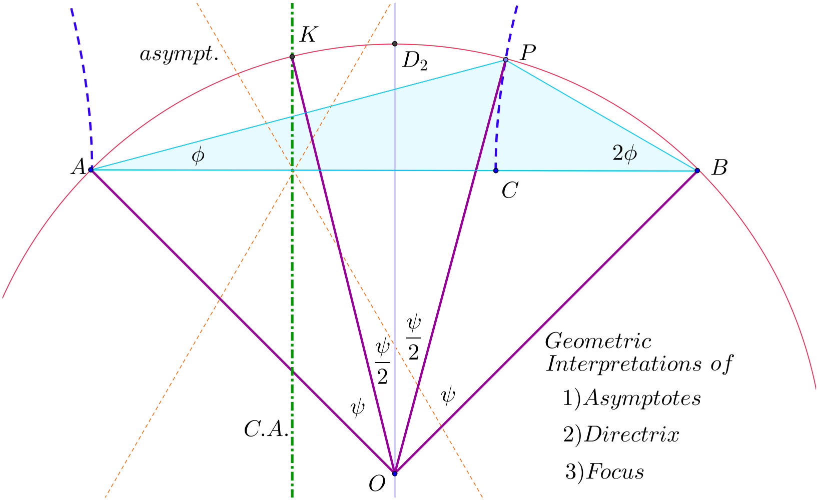
If $\angle PBA = 2\angle PAB$, then $\angle AOP = 2\angle POB$. Hence, $\angle POB$ trisects $\angle AOB$.

Proof

We know that $\frac{1}{2}\psi = \phi$ and $2\phi = \frac{1}{2}\alpha$. Solving for ϕ in both equations and equating them, we arrive at $\frac{1}{2}\psi = \frac{1}{4}\alpha$. So, $2\psi = \alpha$ or $\psi = \frac{1}{2}\alpha$. Notice that $\alpha + \psi = \angle AOB$, so $\angle AOB = 3\psi$ or $\psi = \frac{1}{3}\angle AOB$. Therefore, ψ trisects $\angle AOB$.



Geometric Interpretations of Features of Γ



Geometric Interpretations of

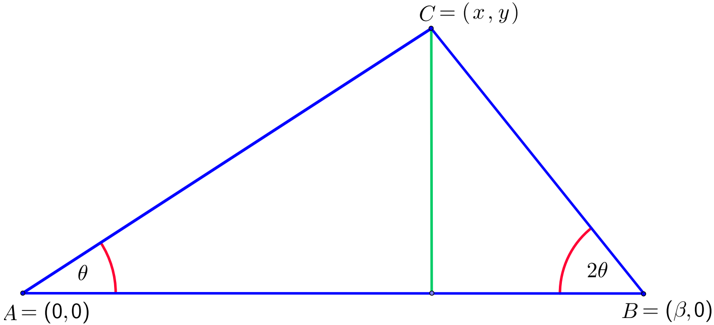
1) *Asymptotes*

2) *Directrix*

3) *Focus*

Investigation of Further Curves

Can we modify this triangle to section angles into however many parts we want just as we have trisected an angle using this triangle?



Investigation of Further Curves

Yes, we can by using this triangle!