Rational approximation of semigroups (and beyond)

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Oliver Heaviside (1850-1925)

“Orthodox Mathematicians, when they cannot find the solution of a problem in a plain agebraical form, are apt to take refuge in a definite integral, and call that the solution.”

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Def: A strongly continuous semigroups is a function $T : [0, \infty) \to \mathscr{L}(X)$ that satisfies

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If T is strongly continuous, then the generator of T is defined by

$$Ax := \lim_{t \rightarrow 0} \frac{T(t)x - x}{t}$$

Cauchy Problem

- Heat Equation

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- Heat Equation
- Wave Equation

Idea

Let r be a rational function such that

- $|r(z)| \leq 1$ when $\text{Re}(z) \leq 0$.
- $r(z) = e^z + O(z^{m+1})$ when $z \to 0$.

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$$\left| r^n\left(\frac{tz}{n}\right) - e^{tz} \right| \;=\; \left| r^n\left(\frac{tz}{n}\right) - (e^{\frac{tz}{n}})^n \right|$$

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Padé

Padé's approximants $r_{j,l}$ of the exponential function are of the form $r_{j,l} = \frac{P_l(z)}{Q_j(z)}$ where

$$P_{j,l}(z) = \sum_{k=0}^{l} \frac{(l+j-k)!l!}{j!k!(l-k)!} z^k$$

and

$$Q_{j,l}(z) = \sum_{k=0}^{j} \frac{(l+j-k)!}{k!(j-k)!} (-z)^k.$$

Hille-Phillips Functional Calculus

$F^\omega := \{f(z) = \int_0^\infty e^{zt} d\alpha(t)$, where $\alpha \in \mathrm{NBV}^\omega$ and $\|\alpha\|_\omega = \int_0^\infty e^{\omega t} d|\alpha|(t)\}$ is an algebra .

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Examples: The constant functions are in F^{ω} for every ω and rational function that are bounded on Re$(z) \leq \omega$ are also in F^{ω}.

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Examples: The constant functions are in F^ω for every ω and rational function that are bounded on Re$(z) \leq \omega$ are also in F^ω.

Theorem: If A is the generator of a strongly continuous semigroup T of type (M, ω) in X, then $\Psi : F^\omega \to \mathscr{L}(X)$ defined by

$$\Psi(f)x := \int_0^\infty T(t)x d\alpha(t)$$

is an homomorphism between the algebras F^ω and $\mathscr{L}(X)$. Moreover, if $\Phi(f) := f(A)$, then $\|f(A)\| \leq \|\alpha\|_\omega$.

Hersh - Kato, Brenner - Thomée.

SIAM J. Num. Anal. '79

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Theorem: If A a strongly continuous semigroup T of type $(M, 0)$, then

$$\left\| r^n\left(\frac{t}{n}A\right) f - T(t)f \right\| \leq Ct^{m+1}\frac{1}{n^m}\|A^{m+1}f\|$$

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So What?

Translation semigroup

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In this way,

$$r^n\left(\tfrac{t}{n}D\right) f(0) = C_{n,0,0} f(0) + \sum_{\substack{1\le i\le s\\1\le j\le nr}} C^n_{n,i,j} R^j\left(b_i, \tfrac{t}{n}D\right) f(0)$$

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Moreover,

$$\left| r^n\left(\frac{t}{n}D\right) f(0) - f(t) \right| \leq C t^{m+1} \frac{1}{n^m} \| f^{(m+1)} \|.$$

Examples

Internet

Thanks

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