Stability in The Sigmoid Beverton-Holt Model

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Abstract

Using the Sigmoid Beverton-Holt Model we pinpoint specific effects on populations when the parameter value is varied. We will be exploring three cases in particular: $\delta = 1$, $\delta < 1$, and $\delta = 2$. The real-world effects of the aforementioned cases will also be explored, specifically the "Allee Effect" which states that a population will tend towards extinction if its density drops below a certain threshold point.

1 Introduction

The Sigmoid Beverton-Holt Model is a discrete-time population model

$$x_{n+1} = \frac{ax_n^\delta}{1 + x_n^\delta}, \quad n = 0, 1, ... \tag{1}$$

where $a, \delta > 0$ and the initial condition $x_0 > 0$. In this model x_n represents the population size (or density) in generation n. The case $\delta = 1$ is well known and studied in the literature and is known as the Beverton-Holt model or Pielou logistic model. The sigmoid Beverton-Holt model has been used in the fishery science to help define optimal fishing rates and solve the problem of diminishing stock sizes. The Sigmoid Beverton-Holt Model is especially suited to this endeavor because it can be adjusted to display instances of multiple equilibiria, specifically the special case of the Allee Effect (also referred to as Depensation).

According to the Allee Effect, a population whose number of individuals has decreased past a certain point will spiral towards extinction with no possibility of succor from intervening measures. This suggests that there exist more than one equilibria in such environments. Although there is significant evidence supporting the existence of multiple equilbria in natural populations, it remains difficult to prove. This is partially due to the fact that none of the models currently used have been able to specify the threshold point (known as Allee Threshold) that separates the favorable domain from the unfavorable domain as far as population size (or density) is concerned.

Based on this concept of depensation "too much fishing" is defined as any amount of fishing that results in the stock population density falling below the lower threshold point. In his article, *A Proposal for a Threshold Stock Size* *and Maximum Fishing Mortality Rate*, Grant G. Thompson attempts to both prove that depensation exists in natural populations and define the elusive lower threshold point for population density. The ultimate goal is to be able to prevent the stock populations from dwindling to the point-of-no-return, thus ensuring the stock's survival and longterm production capacity. Thompson also explores the development of constraints to prevent overfishing and the likely impacts of implementing said constraints on a given fishery.

All attempts to procure a quantitative model that can predict a specific lower threshold point for populations have been unsuccessful up to this point. However, Thompson's article does manage to arrive at three general constraints which may serve to prevent overfishing in the industry.

In this paper we will explore three cases of the Sigmoid Beverton-Holt Model. We will attempt to find the equilibria for each case and confirm the stability of each.

2 Preliminaries

In this section we introduce some definitions and results that will be useful in the arguments that follow.

Definition 1 *$\bar{x}$ is an **equilibrium** of the equation*

$$x_{n+1} = f(x_n), n = 0, 1, ... \tag{2}$$

if

$$\bar{x} = f(\bar{x}).$$

The corresponding solution of $\{\bar{x}_n\}$ such that

$$\bar{x}_n = \bar{x}$$

*is also called a **constant solution** or **steady-state solution**. In such cases we say that $\bar{x}$ is a **fixed point** of the function f.*

Definition 2 *(Stability) (i) The equilibrium point $\bar{x}$ of Eq. (2) is called **(locally) stable** if for every $\epsilon > 0$ there exists $\delta > 0$ such that*

$$|x_0 - \bar{x}| < \delta \quad \textit{implies} \quad |x_n - \bar{x}| < \epsilon \quad \textit{for} \quad n \geq 0.$$

*Otherwise, the equilibrium $\bar{x}$ is called **unstable.***
*(ii) The equilibrium point $\bar{x}$ of Eq. (2) is called **(locally) asymptotically stable** if it is stable and there exists γ such that*

$$|x_0 - \bar{x}| < \gamma \quad \textit{implies} \quad \lim_{n\to\infty} |x_n - \bar{x}| = 0.$$

*(iii) The equilibrium point $\bar{x}$ of Eq. (2) is called **globally asymptotically stable** if for every x_0*

$$\lim_{n\to\infty} |x_n - \bar{x}| = 0.$$

(iv) The equilibrium point $\bar{x}$ of Eq. (2) is called ***globally asymptotically stable relative to a set*** *$S \subset \mathbb{R}$ if it is asymptotically stable, and if for every $x_0 \in S$,*

$$\lim_{n\to\infty} |x_n - \bar{x}| = 0.$$

(v) The equilibrium point $\bar{x}$ of Eq. (2) is said to be an ***attractor with the basin of attraction*** *$S \subset \mathbb{R}$ if*

$$\lim_{n\to\infty} x_n = \bar{x}$$

for every solution with $x_0 \in S$.
(vi) The equilibrium point $\bar{x}$ of Eq. (2) is said to be a ***repeller*** *if there exists $\eta > 0$, such that for all x_0 with $0 < |x_0 - \bar{x}| < \eta$, there exists an integer $N \geq 1$ such that*

$$|x_0 - \bar{x}| \geq \eta.$$

(vii) The equilibrium point $\bar{x}$ of Eq. (2) is called ***(locally) semistable from the right (left)*** *if for every $\epsilon > 0$ there exists $\delta > 0$ such that*

$$0 < x_0 - \bar{x} < \delta \quad (-\delta < x_0 - \bar{x} < 0) \quad \textit{implies} \quad |x_n - \bar{x}| < \epsilon \quad \textit{for} \quad n \geq 0.$$

(viii) The equilibrium point $\bar{x}$ of Eq. (2) is called ***(locally) asymptotically semistable from the right (left)*** *if it is (locally) asymptotically semistable from the right (left) and there exists γ such that*

$$0 \leq x_0 - \bar{x} < \gamma \quad (-\gamma < x_0 - \bar{x} \leq 0) \quad \textit{implies} \quad \lim_{n\to\infty} |x_n - \bar{x}| = 0.$$

The following theorem establishes sufficient conditions for local asymptotic stability:

Theorem 3 *Let $\bar{x}$ be an equilibrium of the difference equation (2) where f is a continuously differentiable function at $\bar{x}$.*
(i) If

$$|f'(\bar{x})| < 1$$

then the equilibrium $\bar{x}$ is ***locally asymptotically stable****.*
(ii) If

$$|f'(\bar{x})| > 1$$

then the equilibrium $\bar{x}$ is ***unstable****.*

In the case when $|f'(\bar{x})| \neq 1$, the equilibrium $\bar{x}$ is called **hyperbolic**. Otherwise it is called **nonhyperbolic**. The next theorem provides sufficient condtions for stability in the case when $f'(\bar{x}) = 1$:

Theorem 4 *Given the difference equation $x_{n+1} = f(x_n)$, where $\bar{x}$ is an equilibrium and f is continuously differentiable three times, if $f'(\bar{x}) = 1$ then we have one of the following:*

1. *If $f''(\bar{x}) \neq 0$, then $\bar{x}$ is semi-stable. More specifically, if $f''(\bar{x}) < 0$, $\bar{x}$ is semistable from the right and if $f''(\bar{x}) > 0$, $\bar{x}$ is semistable from the left.*
2. *If $f''(\bar{x}) = 0$, and $f'''(\bar{x}) > 0$, $\bar{x}$ is unstable.*
3. *If $f''(\bar{x}) = 0$, and $f'''(\bar{x}) < 0$, $\bar{x}$ is locally asymptotically stable.*

The next theorem is a well known result about the convergence of monotone sequences.

Theorem 5 *If a sequence $\{x_n\}$ is monotonic (increasing or decreasing) and bounded, then the sequence $\{x_n\}$ converges.*

3 The case $\delta = 1$

We begin with the Beverton-Holt Model (also known as Pielou Logistic Model),

$$x_{n+1} = \frac{ax_n}{1 + x_n}, \quad n = 0, 1, ... \tag{3}$$

To find equilibria we need to solve the equation

$$x = \frac{ax}{1 + x}$$

or

$$x(1 + x) = ax.$$

Factoring reveals two solutions for x:

$$\begin{aligned} x &= 0, \\ x &= a - 1. \end{aligned}$$

Thus, 0 and $a - 1$ are both equilibrium points of the model

$$x_{n+1} = \frac{ax_n}{1 + x_n}.$$

Regardless of the values of a, 0 will always be an equilibirum point for this model. However, the situation with the equilibrium $a - 1$ is more complicated. We will explore three cases, namely: $0 < a < 1$, $a > 1$, and $a = 1$. In the cases where $0 < a < 1$ and $a = 1$ we find that the equilibrium $x = a - 1 \leq 0$ so it will be excluded.

The above considerations can be summarized in the following lemma:

Lemma 6 *Consider the Beverton-Holt model (3) where $a > 0, x_0 > 0$. Then the following statements are true:*
(i) If $0 < a \leq 1$, then 0 is the only equilibrium of the equation.
(ii) If $a > 1$, then the equation has two equilibria: 0 and the positive equilibrium point $\bar{x} = a - 1$.

Next, we will study the stability character of the 0 equilibrium. Let

$$f(x) = \frac{ax}{1+x},$$

then

$$f'(x) = \frac{(1+x)(a) - (ax)(1)}{(1+x)^2} = \frac{a}{(1+x)^2}.$$

Since

$$|f'(0)| = a \begin{cases} < 1 & \text{if} \quad a < 1 \\ = 1 & \text{if} \quad a = 1 \\ > 1 & \text{if} \quad a > 1. \end{cases}$$

according to Theorem 3 it follows that 0 is locally asymptotically stable if $0 < a < 1$ and unstable in the case $a > 1$.

In the case when $a = 1$, we have $f'(0) = 1$ and we will use Theorem 4. In this case

$$f''(x) = -\frac{2}{(1+x)^3}$$

and we have

$$f''(0) = -2 < 0$$

so 0 is semi-stable from the right. Since we consider only positive solutions, we may say that 0 is stable.

Again, we summarize the previous consideration in the following lemma:

Lemma 7 *Consider the Beverton-Holt model (3) where $a > 0, x_0 > 0$. Then the following statements are true:*
(i) If $0 < a < 1$, then 0 is locally asymptotically stable.
(ii)If $a = 1$, then 0 is locally stable.
(iii) If $a > 1$, then 0 is unstable.

In order to be considered globally asymptotically stable and equilibrium must meet the following two criteria:

1. Equilibrium must be locally asymptotically stable.

2. Equilibirum must attract all non-negative solutions to itself.

So, in order to establish the global asymptotic stability of the 0 equlibrium in the case $0 < a \leq 1$ we need to prove global attractivity. our model.To do this we use Theorem 5. We will prove the convergence of the sequence $\{x_n\}$ by proving its monotonicity and showing that it is bounded. We will now prove that the sequence is decreasing by showing that

$$x_{n+1} - x_n < 0.$$

Using the recursion equation for this model we can set the left side of the above equation equal to the right side of our given model:

$$x_{n+1} - x_n = \frac{ax_n}{1+x_n} - x_n = x_n(\frac{a-1-x_n}{1+x_n}) < 0$$

because $a \leq 1$ and our sequence $\{x_n\}$ contains only positive values. Therefore the sequence $\{x_n\}$ is monotonically decreasing. Also the sequence $\{x_n\}$ is positive and

$$0 < ...x_n < x_{n-1} < ... < x_1 < x_0$$

so it is bounded. Therefore according the Theorem 5 it converges. Furthermore, because $\{x_n\}$ converges, there exists a non-negative real number x, such that

$$\lim_{n\to\infty} x_n = x.$$

Taking the limit of both sides of our recursion equation,

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} (\frac{ax_n}{1+x_n})$$

and executing one algebraic step,

$$x = \frac{ax}{1+x}$$

we arrive at the equation $x(1+x) = ax$, where the only non-negative solution is $x = 0$.

So for $0 < a < 1$ the equilibirum does attract all positive solutions. Therefore we have the following theorem:

Theorem 8 *Consider the Beverton-Holt model (3) where* $0 < a \leq 1, x_0 > 0$. *Then the given model has only one equlibrium* 0 *which is globally asymptotically stable.*

Next we will focus on the case when $a > 1$. We already know that our model has two equilibria: unstable 0 equilibrium and positive equilibrium $\bar{x} = a - 1$.

Next we test the stability of our second equilibrium, $x = a - 1$, by

$$|f'(a-1)| = \frac{1}{a} < 1$$

This verifies that the equilibrium $\bar{x} = a - 1$ is locally asymptotically stable.

Also recall that in order to be considered globally asymptotically stable an equilibrium must be both locally asymptotically stable and be an attractor of all positive solutions. Thus we will once again apply Theorem 5 to examine the global asymptotic stability of our equilibrium $x = a - 1$ by proving its monotonicity and showing that it is bounded.

We begin by letting $0 < x_0 \leq a - 1$, and showing that $0 < x_n \leq a - 1$ for all $n > 0$. It can be shown that this is true using a simple induction argument. Namely, since the function $f(x) = \frac{ax}{1+x}$ is increasing we have

$$0 < x_{n+1} = f(x_n) = \frac{ax_n}{1 + x_n} < \frac{a(a-1)}{1 + (a-1)} = a.$$

So by induction we may conclude that $0 < x_n \leq a - 1$ for all $n > 0$, and the sequence $\{x_n\}$ is bounded.

Next we have

$$x_{n+1} - x_n = x_n(\frac{a - 1 - x_n}{1 + x_n}) > 0.$$

So the sequence $\{x_n\}$ is increasing and bounded by 0 and $a - 1$, thus, it converges to a positive limit x. Taking the limit of both sides of the equation it follows that that $x = a - 1$, and all solutions with initial conditions satisfying $0 < x_0 < a - 1$ converge to $a - 1$.

By a similar argument it can be shown that if $x_0 > a - 1$ the corresponding solution $\{x_n\}$ is decreasing and converges to $a-1$. So $x = a-1$ attracts all positive solutions and is thus globally asmyptotically stable. Again we summarize the previous consideration in the following theorem

Theorem 9 *Consider the Beverton-Holt model (3) where $a > 1, x_0 > 0$. Then the given model has two equilibria: ustable 0 equlibrium and globally asymptotically stable positive equilibrium $\bar{x} = a - 1$.*

4 The Case $\delta < 1$

In the case $\delta < 1$, we want to examine the existence and stability of equilibria of the difference equation

$$x_{n+1} = \frac{ax^\delta}{1 + x_n{}^\delta}, \quad n = 0, 1, ...$$

where $\delta < 1, a > 0$, and $x_0 > 0$.

To find the equilibria we need to solve the equation

$$x = \frac{ax^\delta}{1 + x^\delta}$$

or

$$(1 + x^\delta)x = ax^\delta.$$

Clearly 0 is always a solution. To see if there are other solutions we divide both sides by x^δ and obtain

$$x^{1-\delta} + x = a$$

or

$$x^{1-\delta} + x - a = 0.$$

Now let

$$g(x) = x^{1-\delta} + x - a.$$

Then

$$g'(x) = (1-\delta)x^{-\delta} + 1,$$

so for all $x > 0$, $g'(x) > 0$ and therefore $g(x)$ is an increasing and continuous function.

Since g is a continuous real valued function we can apply the Intermediate Value Theorem to prove that there exists a unique solution where there is an equilibrium other than 0. Since $g(0) = -a < 0$ and $g(\infty) = \lim_{x\to\infty} g(x) = \infty$ it follows that there exists $B > 0$, such that $g(B) > 0$ and $g(x) > 0$ for $x > B$. So by applying the Intermediate Value Theorem we obtain that there exists a unique solution of $g(x) = 0$ and that is our positive equilibrium $\bar{x}$.

Since we know that there exists a unique positive equilibrium $\bar{x}$ we will examine its stability. To do this we consider the function

$$f(x) = \frac{ax^\delta}{1+x^\delta}$$

and find $f'(x)$:

$$f'(x) = \frac{(1+x^\delta)\delta a x^{\delta-1} - ax^\delta(\delta x^{\delta-1}}{(1+x^\delta)^2} = \frac{\delta a x^{\delta-1} + \delta a x^{2\delta-1} - \delta a x^{2\delta-1}}{(1+x^\delta)^2} = \frac{\delta a x^{\delta-1}}{(1+x^\delta)^2}.$$

So

$$f'(\bar{x}) = \frac{\delta a \bar{x}^{\delta-1}}{(1+\bar{x}^\delta)^2}.$$

Since

$$1 + \bar{x}^\delta = a\bar{x}^{\delta-1}$$

we have

$$f'(\bar{x}) = \frac{\delta a \bar{x}^{\delta-1}}{(ax^{\delta-1})^2} = \frac{\delta}{a\bar{x}^{\delta-1}}.$$

Since $f'(\bar{x}) > 0$, the condition for local asymptotic stability

$$|f'(\bar{x})| < 1$$

is equivalent to

$$\delta < a\bar{x}^{\delta-1}$$

or

$$(\frac{\delta}{a})^{\frac{1}{\delta-1}} > \bar{x}.$$

Since g is an increasing function the above condition is equivalent to

$$g((\frac{\delta}{a})^{\frac{1}{\delta-1}}) > g(\bar{x}) = 0.$$

Since

$$
\begin{aligned}
g((\frac{\delta}{a})^{\frac{1}{\delta-1}}) &= \left((\frac{\delta}{a})^{\frac{1}{\delta-1}}\right)^{1-\delta} + (\frac{\delta}{a})^{\frac{1}{\delta-1}} - a \\
&= \frac{a}{\delta} - a + (\frac{\delta}{a})^{\frac{1}{\delta-1}} \\
&= \frac{a(1-\delta)}{\delta} + (\frac{\delta}{a})^{\frac{1}{\delta-1}} > 0,
\end{aligned}
$$

We know $(\frac{\delta}{a})^{\frac{1}{\delta-1}} > \bar{x}$, which implies $f'(x) < 1$. Therefore we have shown the equilibrium $\bar{x}$ is locally asymptotically stable.

Regarding 0 equilibrium we obtain that $f'(0)$ is undefined, more precisely $f'(0+) = \infty$ and the 0 equilibrium is unstable.

We now proceed to test our positive equilibrium $\bar{x}$ for global asymptotic stability. Since

$$
f'(x) = \frac{\delta a x^{\delta-1}}{(1+x^{\delta})^2} > 0
$$

the function f is increasing. Next, we examine the fitness function $\frac{f(x)}{x}$. Thus we arrive at the following:

$$
\frac{f(x)}{x} = \frac{a x^{\delta-1}}{1+x^{\delta}} = \frac{a}{x^{1-\delta}(1+x^{\delta})}.
$$

Since both exponents are positive we see that the fitness function is decreasing. As a result, if $x < \bar{x}$ then $\frac{f(x)}{x} > \frac{f(\bar{x})}{\bar{x}}$. However, we also know that $f(\bar{x}) = \bar{x}$, since $\bar{x}$ represents equilibrium. This means that $\frac{f(x)}{x} > 1 \Rightarrow f(x) > x$ when $x < \bar{x}$. Similarly, $f(x) < x$ if $x > \bar{x}$.

Now we want to examine the case when $x_0 < \bar{x}$. To show that $x_n < \bar{x}$ for all n we will use induction. We know $x_0 < \bar{x}$ by assumption. So assume $x_k < \bar{x}$ for some $k > 0$, and look at x_{k+1}. Now

$$
x_{k+1} = f(x_k) < f(\bar{x}) = \bar{x}
$$

because $x_k < \bar{x}$ and f is an increasing function. Therefore $x_{k+1} < \bar{x}$ and the result holds by induction.

To show monotonicity we use our standard test: $x_{n+1} - x_n$ where if it is greater than 0 then the sequence is increasing and, similarly, if it is less than 0 then it is decreasing. So

$$
x_{n+1} - x_n = f(x_n) - x_n > 0
$$

since $f(x) > x$ when $x < \bar{x}$. Therefore we have a sequence $\{x_n\}$ that is bounded and increasing monotonically. So $\{x_n\}$ converges to a positive limit x.

Lastly, we take the fact that $x_{n+1} = \frac{a{x_n}^{\delta}}{1+{x_n}^{\delta}}$ and take the limit of both sides; which results in the equation $x = \frac{ax^{\delta}}{1+x^{\delta}}$. At this point we know that the only non-zero solution to this equation is $\bar{x}$. So $\bar{x}$ attracts all solutions for $x < \bar{x}$. By a similar argument we can show that $\bar{x}$ attracts all solutions for $x > \bar{x}$, thus $\bar{x}$ is globally asymptotically stable.

So we summarize the previous consideration in the following result:

Theorem 10 *Consider the Sigmoid Beverton-Holt model (1) where* $0 < \delta < 1, a > 0, x_0 > 0$. *Then the model has two equilibria: unstable* 0 *equilibrium and globally asymptotically stable positive equilibrium* $\bar{x}$.

5 The Case $\delta = 2$

Here we have the case where $\delta = 2$. So our model is

$$x_{n+1} = \frac{a{x_n}^2}{1 + {x_n}^2}. \tag{4}$$

To find equilibria we need to solve the equation:

$$x = \frac{ax^2}{1 + x^2}.$$

In order to find its equilibrium we must solve this equation for x. Doing some simple algebra we determine the equilibria to be $x = 0$ and solutions to the equation $x^2 - ax + 1 = 0$. We employ the quadratic formula to find:

$$x = \frac{a \pm \sqrt{a^2 - 4}}{2}.$$

From looking at the determinant of the quadratic equation we see that there are three distinct cases for our model when $\delta = 2$:

1. If $a < 2$, the model has only 0 as an equilibrium.
2. If $a = 2$, the model has two equilibria: the 0 equlibrium and the positive equilibrium $\bar{x} = 1$.
3. If $a > 2$ the model has three equilibria: the 0 equilibrium and two positive equilibria $\tilde{x} = \frac{a-\sqrt{a^2-4}}{2}$ and $\bar{x} = \frac{a+\sqrt{a^2-4}}{2}$.

Beginning with the case where $a < 2$ we test the stability of the only equilibrium, $x = 0$, by plugging it into the derivative of the original function.

$$\begin{aligned} f'(x) &= \frac{(1 + x^2)2ax - ax^2(2x)}{(1 + x^2)^2} \\ &= \frac{2ax + 2ax^3 - 2ax^3}{(1 + x^2)^2} \\ &= \frac{2ax}{(1 + x^2)^2}, \end{aligned}$$

which simplifies as follows:

$$f'(0) = \frac{0}{1} = 0.$$

So $x = 0$ is locally asymptotically stable.

Now we will show that the sequence $\{x_n\}$ converges. However, to do so we must first show that $\{x_n\}$ is decreasing monotonically. We will show this by proving $x_{n+1} - x_n < 0$.

$$x_{n+1} - x_n = \frac{a{x_n}^2}{1+{x_n}^2} - x_n = \frac{a{x_n}^2 - x_n(1+{x_n}^2)}{1+{x_n}^2} = x_n(\frac{ax_n - 1 - {x_n}^2}{1+{x_n}^2})$$

Since the sequence $\{x_n\}$ is always positive, the only part of the equation in question is $h(x_n) = -{x_n}^2 + ax_n - 1$, which represents a downward opening parabola. Therefore, if the y-value of the vertex of the parabola is less than 0 the expression is always negative.

Since $-{x_n}^2 + ax_n - 1$ has its vertex at the point $(-\frac{b}{2a}, h(\frac{b}{2a}))$, we can simply plug in values to find whether the vertex lies below or above the x-axis:

$$(\frac{-a}{-2}, h(\frac{-a}{-2})) = (\frac{a}{2}, -1).$$

Thus the vertex is found to lie below the x-axis, meaning the expression $-x_n^2 + ax_n - 1$ is always negative. So $x_n(\frac{-{x_n}^2 + ax_n - 1}{1+x^2}) < 0$. Therefore the sequence $\{x_n\}$ is decreasing and bounded with $0 < x_n < x_0$ for all $n > 0$.

Since $\{x_n\}$ is bounded and decreasing then, by Theorem 5, it converges. So there is a nonnegative x such that

$$\lim_{n\to\infty} x_n = x.$$

Then,

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} (\frac{a{x_n}^2}{1+{x_n}^2}) \Rightarrow x = \frac{ax^2}{1+x^2}$$

Since $a < 2$, $x = 0$. So it follows that the equilibrium $x = 0$ is globally asymptotically stable.

We summarize the above consideration in the following theorem:

Theorem 11 *Consider the Sigmoid Beverton-Holt model (4) where* $0 < a < 2, x_0 > 0$. *Then* 0 *is the only equilibrium and it is globally asymptotically stable.*

Next we have the case where $a = 2$. The model has two equilibria, 0 and $\bar{x} = 1$. Since

$$f'(x) = \frac{2ax}{(1+x^2)^2}$$

we have

$$|f'(0)| = 0 < 1,$$

so 0 is locally asymptotically stable. Furthermore

$$f'(1) = \frac{2a}{4} = \frac{4}{4} = 1$$

so we need to use Theorem 4. Then

$$\begin{aligned} f''(x) &= \frac{(1+x^2)^2 2a - 2ax(2(1+x^2)2x)}{(1+x^2)^4} = \frac{2a(1-3x^2)}{(1+x^2)^3} \\ f''(1) &= \frac{-3a}{4} < 0 \end{aligned}$$

Since $f''(1) < 0$, then $\bar{x} = 1$ is unstable, but it is semi-stable from the right. So now we will show, using the Monotonic Convergence Theorem, that neither $x = 0$, or $x = 1$ is globally asymptotically stable in this case.

We begin by showing that when $0 < x_0 < \bar{x}$ the sequence $\{x_n\}$ converges.

$$x_{n+1} - x_n = \frac{ax_n^2}{1+x_n^2} - x_n$$

From an argument already stated we know that the sequence $\{x_n\}$ is non-increasing. So $\{x_n\}$ is bounded and monotonic, therefore it converges to 0.

By a similar argument we can conclude that when $x_0 > \bar{x}$, $\{x_n\}$ converges to $\bar{x}$. So we have the following result:

Theorem 12 *Consider the Sigmoid Beverton-Holt model (4) where* $a = 2, x_0 > 0$. *Then the following statements are true:*
(i) The model has two equilibria 0 *and* 1.
(ii) 0 *is locally asymptotically stable with basin of attraction* $[0, 1)$.
(iii) 1 *is locally asymoptotically semistable from the right with the basin of attraction* $(1, \infty)$.

Finally we will examine the case when $a > 2$ and $\delta = 2$. Recall that we have three equilibria 0, $\tilde{x}$ and $\bar{x}$ where

$$\tilde{x} = \frac{a - \sqrt{a^2-4}}{2} \text{ and } \bar{x} = \frac{a + \sqrt{a^2-4}}{2}.$$

We find the stability of the equilibria as follows:

$$f'(x) = \frac{2ax(1+x^2) - 2x(2ax)}{(1+x^2)^2} = \frac{2ax}{(1+x^2)^2}.$$

Each equilibrium must be substituted in for the x value.

$$\begin{aligned} f'(0) &= \frac{2a(0)}{(1+0^2)^2} = \frac{0}{1} = 0 < 1 \\ f'(\bar{x}) &= \frac{2a\bar{x}}{(1+\bar{x}^2)^2} = \frac{2a\bar{x}}{(a\bar{x})^2} = \frac{2}{a\bar{x}} = \frac{2}{a(\frac{a+\sqrt{a^2-4}}{2})} = \frac{4}{a^2 + a\sqrt{a^2-4}} < 1 \\ f'(\tilde{x}) &= \frac{2a\tilde{x}}{(1+\tilde{x}^2)^2} = \frac{2a\tilde{x}}{(a\tilde{x})^2} = \frac{2}{a\tilde{x}} = \frac{2}{a(\frac{a-\sqrt{a^2-4}}{4})} = \frac{4}{a^2 - a\sqrt{a^2-4}} > 1 \end{aligned}$$

The results show that 0 and $\bar{x}$ are both locally stable equilibria. On the other hand, $\tilde{x}$ is unstable.

Next, we will show that when $x_0 < \tilde{x}$, $x_n < \tilde{x}$ for all $n > 0$. We know $x_0 < \tilde{x}$ by assumption, so assume $x_k < \tilde{x}$ and look at x_{k+1}:

$$x_{k+1} = \frac{a{x_k}^2}{1 + {x_k}^2} < \frac{a\tilde{x}^2}{1 + \tilde{x}^2} = \tilde{x}.$$

Thus when $x_0 < \tilde{x}$, $x_n < \tilde{x}$ for all $n > 0$ by induction. Now we will examine the monotonicity of $\{x_n\}$ by looking at $x_{n+1} - x_n$:

$$x_{n+1} - x_n = \frac{a{x_n}^2}{1 + {x_n}^2} - x_n = x_n(\frac{ax_n - 1 - x_n^2}{(1 + x_n^2)})$$

When $x_n < \tilde{x}$, $ax_n^2 - 1 - x_n^2 < 0$, so $x_{n+1} - x_n < 0$. Thus $\{x_n\}$ is decreasing. By Theorem 5 $\{x_n\}$ converges and, since $\{x_n\}$ decreases and $x_n < \tilde{x}$, it must converge to 0.

Next we will look at the case where $\tilde{x} < x_0 < \bar{x}$, we will show that $\tilde{x} < x_n < \bar{x}$ for all $n > 0$. Assume that the result holds for x_k and look at x_{k+1}.

$$x_{k+1} = \frac{a{x_k}^2}{1 + {x_k}^2} > \frac{a\tilde{x}^2}{1 + \tilde{x}^2} = \tilde{x}$$

$$x_{k+1} = \frac{a{x_k}^2}{1 + {x_k}^2} < \frac{a\bar{x}^2}{1 + \bar{x}^2} = \bar{x}x_{n+1} - x_n = \frac{a{x_n}^2}{1 + {x_n}^2} - x_n = x_n(\frac{ax_n - 1 - {x_n}^2}{1 + {x_n}^2}).$$

So $\tilde{x} < x_n < \bar{x}$ for all $n > 0$. Now we will examine the monotonicity of the sequence $\{x_n\}$, where $ax_n - 1 - {x_n}^2 > 0$ when $\tilde{x} < x_n < \bar{x}$. This shows that $\{x_n\}$ increases and is attracted to $\bar{x}$.

By an argument similar to the one used for $x_0 < \tilde{x}$, we can show that when $x_0 > \bar{x}$ the sequence $\{x_n\}$ converges to $\bar{x}$. Thus, $\bar{x}$ is locally asymptotically stable. So we have the following result:

Theorem 13 *Consider the Sigmoid Beverton-Holt model (4) where* $a > 2, x_0 > 0$. *Then the following statements are true:*
(i) The model has three equilibria 0 *and two positive equilibria* $\tilde{x} = \frac{a-\sqrt{a^2-4}}{2}$ *and* $\bar{x} = \frac{a+\sqrt{a^2-4}}{2}$.
(ii) 0 *is locally asymptotically stable with basin of attraction* $[0, \tilde{x})$.
(iii) $\tilde{x}$ *is ustable.*
(iii) $\bar{x}$ *is locally asymoptotically stable with basin of attraction* $(\tilde{x}, \infty)$.

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