The Sigmoid Beverton-Holt Model

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General Definition

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What exactly *is* the Sigmoid Beverton-Holt Model?

A discrete-time population model that uses a function of the number of individuals in the present generation to provide the expected population density for subsequent generations.

Applications

How does the SBHM apply to the real-world?

The SBHM can be used to:

1. Define optimal fishing rates to prevent diminishing stock sizes in the fishery industry.
2. Foresee the extinction of a given natural population.
3. Calculate insurance costs.
4. Estimate the future population density of a given natural population.
5. Predict most future trends in natural populations.

Overview

What are we going to *do* with the SBHM?

Let's take a look at what happens when the parameter value, δ, is varied. We explored three cases in particular:

$$\delta = 1$$

$$\delta < 1$$

$$\delta = 2$$

Let's take it step-by-step, using the case $\delta = 1$ as our example.

The $\delta = 1$ Case

Step 1: Set Value for δ

Start with the SBHM, $x_{n+1} = \frac{a{x_n}^{\delta}}{1+{x_n}^{\delta}}$, and set a value for δ.

Here we have chosen $\delta = 1$ where $0 < a < 1$, as shown below:

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$$x_{n+1} = \frac{a{x_n}^{\delta}}{1 + {x_n}^{\delta}} \Rightarrow x_{n+1} = \frac{ax_n}{1 + x_n}$$

when $\delta = 1$.

IMPORTANT KEY TERM!

Equilibrium

A condition in which all acting influences are cancelled by others, resulting in a stable, balanced, or unchanging system.

The $\delta = 1$ Case

Step 2: Simplify and Solve for x

Simplify the equilibrium equation and solve for x to find the equilibria.

$$x = \frac{ax}{1+x} \Rightarrow x(1+x) = ax$$

Which results in two equilibria: $x = 0$ $x = a - 1$

Note: Since we are looking at instances where $0 < a < 1$, we will discard $x = a - 1$ because it does not provide a positive solution.

IMPORTANT KEY TERM!

Local Asymptotic Stability

If the result of plugging the equilibrium value into the derivative of the equilibrium equation is between -1 and 1, then it is L.A.S.

The $\delta = 1$ Case

Step 3: Test for Local Asymptotic Stability

Now we test the remaining equilibrium, $x = 0$, for L.A.S.

First we find the derivative of the equilibrium equation:

$$f'(x) = \frac{a}{(1+x)^2}.$$

Then we plug in our equilibrium value of 0:

$$f'(0) = \frac{a}{1}$$

The result is less than 1 for all $0 < a < 1$. Since this falls between -1 and 1 the equilibrium is L.A.S.

IMPORTANT KEY TERM!

Global Asymptotic Stability

If the sequence is shown to be both bounded and increasing/decreasing monotonically, then it is known to be G.A.S.

The $\delta = 1$ Case

Step 4: Test for Global Asymptotic Stability

Since $x = 0$ is L.A.S., we can now test it for G.A.S. by proving convergence.

To prove convergence two criteria must be satisfied:

1. Sequence must be increasing or decreasing monotonically
2. Sequence must be bounded

IMPORTANT KEY TERM!

Monotonic Convergence Theorem

If a sequence $\{x_n\}$ is monotonic (increasing or decreasing) and bounded, then the sequence $\{x_n\}$ converges.

The $\delta = 1$ Case

Step 5: Prove Monotonicity

To prove monotonicity we set our recursion equation equal to our model and solve:

$$x_{n+1} - x_n = \frac{ax_n}{1 + x_n} - x_n$$

The resulting expression, $x_n(\frac{a-1-x_n}{1+x_n})$, is negative; so the sequence is decreasing monotonically.

The $\delta = 1$ Case

Step 6: Prove the Sequence is Bounded

Since the sequence $\{x_n\}$ is decreasing $(x_{n+1} < x_n < ... < x_2 < x_1 < x_0)$, and x_m is greater than 0 for all m, we arrive at the following inequality:

$$0 < x_m \leq x_0.$$

Which tells us that the sequence is bounded by 0 and x_0.

The sequence is both monotonic and bounded, therefore it is G.A.S.

The $\delta = 1$ Case

Graph: $\delta = 1$, $a = 1$, initial value $= .35$

The $\delta = 1$ Case

Graph: $\delta = 1$, $a = 1$, initial value $= 1$

The $\delta = 1$ Case

The $\delta = 1$ Case

Summation of $\delta = 1$ Case

We have now shown our equilibrium, $x = 0$, to be not only L.A.S. but also G.A.S. for our model when $\delta = 1$ where $0 < a < 1$. The same step-by-step process just used in our $\delta = 1$ case was also employed in our other two cases, where $\delta < 1$ and $\delta = 2$.

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