The Sigmoid Beverton-Holt Model

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August 24, 2010
What exactly is the Sigmoid Beverton-Holt Model?

A discrete-time population model that uses a function of the number of individuals in the present generation to provide the expected population density for subsequent generations.

\[ x_{n+1} = \frac{ax_n^\delta}{1 + x_n^\delta}. \]
The Sigmoid Beverton-Holt Model

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Applications

How does the SBHM apply to the real-world?

The SBHM can be used to:

1. Define optimal fishing rates to prevent diminishing stock sizes in the fishery industry.
2. Foresee the extinction of a given natural population.
3. Calculate insurance costs.
4. Estimate the future population density of a given natural population.
5. Predict most future trends in natural populations.
What are we going to do with the SBHM?

Let’s take a look at what happens when the parameter value, $\delta$, is varied. We explored three cases in particular:

- $\delta = 1$
- $\delta < 1$
- $\delta = 2$

Let’s take it step-by-step, using the case $\delta = 1$ as our example.
Step 1: Set Value for $\delta$

Start with the SBHM, $x_{n+1} = \frac{ax_n^\delta}{1 + x_n^\delta}$, and set a value for $\delta$.

Here we have chosen $\delta = 1$ where $0 < a < 1$, as shown below:

$$x_{n+1} = \frac{ax_n^\delta}{1 + x_n^\delta}$$
The $\delta = 1$ Case

**Step 1: Set Value for $\delta$**

Start with the SBHM, $x_{n+1} = \frac{ax_n^\delta}{1 + x_n^\delta}$, and set a value for $\delta$.

Here we have chosen $\delta = 1$ where $0 < a < 1$, as shown below:

$$x_{n+1} = \frac{ax_n^\delta}{1 + x_n^\delta} \Rightarrow x_{n+1} = \frac{ax_n}{1 + x_n}$$

when $\delta = 1$. 

Equilibrium

A condition in which all acting influences are cancelled by others, resulting in a stable, balanced, or unchanging system.
Step 2: Simplify and Solve for $x$

Simplify the equilibrium equation and solve for $x$ to find the equilibria.

$$x = \frac{ax}{1 + x} \Rightarrow x(1 + x) = ax$$

Which results in two equilibria: $x = 0$ $x = a - 1$

**Note:** Since we are looking at instances where $0 < a < 1$, we will discard $x = a - 1$ because it does not provide a positive solution.
Local Asymptotic Stability

If the result of plugging the equilibrium value into the derivative of the equilibrium equation is between -1 and 1, then it is L.A.S.
Now we test the remaining equilibrium, \( x = 0 \), for L.A.S.

First we find the derivative of the equilibrium equation:

\[
f'(x) = \frac{a}{(1 + x)^2}.
\]

Then we plug in our equilibrium value of 0:

\[
f'(0) = \frac{a}{1}
\]

The result is less than 1 for all \( 0 < a < 1 \). Since this falls between \(-1\) and 1 the equilibrium is L.A.S.
Global Asymptotic Stability

If the sequence is shown to be both bounded and increasing/decreasing monotonically, then it is known to be G.A.S.
Step 4: Test for Global Asymptotic Stability

Since \( x = 0 \) is L.A.S., we can now test it for G.A.S. by proving convergence.

To prove convergence two criteria must be satisfied:

1. Sequence must be increasing or decreasing monotonically
2. Sequence must be bounded
Monotonic Convergence Theorem

If a sequence $\{x_n\}$ is monotonic (increasing or decreasing) and bounded, then the sequence $\{x_n\}$ converges.
Step 5: Prove Monotonicity

To prove monotonicity we set our recursion equation equal to our model and solve:

\[ x_{n+1} - x_n = \frac{ax_n}{1 + x_n} - x_n \]

The resulting expression, \( x_n\left(\frac{a-1-x_n}{1+x_n}\right) \), is negative; so the sequence is decreasing monotonically.
Step 6: Prove the Sequence is Bounded

Since the sequence \( \{x_n\} \) is decreasing \((x_{n+1} < x_n < \ldots < x_2 < x_1 < x_0)\), and \(x_m\) is greater than 0 for all \(m\), we arrive at the following inequality:

\[ 0 < x_m \leq x_0. \]

Which tells us that the sequence is bounded by 0 and \(x_0\).

The sequence is both monotonic and bounded, therefore it is G.A.S.
The $\delta = 1$ Case

Graph: $\delta = 1$, $a = 1$, initial value $= .35$
The $\delta = 1$ Case

Graph: $\delta = 1$, $a = 1$, initial value = 1
The $\delta = 1$ Case

We have now shown our equilibrium, \( x = 0 \), to be not only L.A.S. but also G.A.S. for our model when \( 0 < a < 1 \). The same step-by-step process just used in our $a = 1$ case was also employed in our other two cases, where $a < 1$ and $a = 2$. 
The $\delta = 1$ Case

**Summation of $\delta = 1$ Case**

We have now shown our equilibrium, $x = 0$, to be not only L.A.S. but also G.A.S. for our model when $\delta = 1$ where $0 < a < 1$. The same step-by-step process just used in our $\delta = 1$ case was also employed in our other two cases, where $\delta < 1$ and $\delta = 2$. 


The End