# Ricker's Population Model

## The Study of the Existence and Stability of Equilibria Within an Ecosystem

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# Outline

1. Motivation
   - History
   - Applications
2. Preliminaries
   - Necessary Definitions
   - Necessary Theorems
3. Examples and Experimentations
   - 2 Arbitrary Cases
4. Conclusion

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# The Originator

## Dr. Bill Ricker (1908-2001)

$$x_{n+1} = {x_n}^{\delta} e^{r - x_{n-k}} \tag{1}$$

$$r, \delta, x_0 \in \mathbb{R}^+, n \in \mathbb{Z}^+$$

Best known for the Ricker model, which he developed in his studies of stock and recruitment in fisheries.

Throughout his lifetime, Dr. Ricker published 296 papers and books, 238 translations, and 148 scientific and literary manuscripts."

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# The Applications

- Fishery Sciences
- Biological Sciences
- Human Population Modeling

Any field of science that involves a population of species, Ricker's model can be applied. However, these are only a few of the areas of which Ricker's can be used. Day after day, more applications are being developed for this model.

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# Definitions

## Definition

$\bar{x}$ is an **equilibrium** of the equation

$$x_{n+1} = f(x_n), n = 0, 1, ... \quad (2)$$

if

$$\bar{x} = f(\bar{x}).$$

The correpsonding solution $\{\bar{x}_n\}$ such that

$$\bar{x}_n = \bar{x}$$

is called also a **constant solution** or **steady-state solution.** Also, in such cases we say that $\bar{x}$ is a **fixed point** of the function $f$.

# Definitions

## Definition

(Stability) (i) The equilibrium point $\bar{x}$ of Eq. (2) is called **(locally) stable** if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$|x_0 - \bar{x}| < \delta \quad \text{implies} \quad |x_n - \bar{x}| < \epsilon \quad \text{for} \quad n \geq 0.$$

Otherwise, the equilibrium $\bar{x}$ is called **unstable.**
(ii) The equilibrium point $\bar{x}$ of Eq. (2) is called **(locally) asymptotically stable (LAS)** if it is stable and there exists $\gamma$ such that

$$|x_0 - \bar{x}| < \gamma \quad \text{implies} \quad \lim_{n\to\infty} |x_n - \bar{x}| = 0.$$

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# Theorems

## Theorem

*Let $\bar{x}$ be an equilibrium of the difference equation (1) where $f$ is a continuously differentiable function at $\bar{x}$.*
*(i) If*

$$\left|f'(\bar{x})\right| < 1$$

*then the equilibrium $\bar{x}$ is* ***locally asymptotically stable****.*
*(ii) If*

$$\left|f'(\bar{x})\right| > 1$$

*then the equilibrium $\bar{x}$ is* ***unstable****.*

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# 2 Arbitrary Cases

For the two particular cases we will present, we will briefly describe what will happen when:

- $0 < \delta \leq 1$
- $\delta > 1$

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# Set Up

In order to better understand the difference equation

$$x_{n+1} = {x_n}^{\delta} e^{r-x_n},$$

we represent the model with the function

$$f(x) = x^{\delta} e^{r-x}.$$

We can use this function to better solve for equilibria and analyze stability.

# Finding Equilibria

In order to solve for equilibria of our equation, we must set the function $f(x) = x$. So

$$x^{\delta} e^{r-x} = x$$

$$x^{\delta} e^{r-x} - x = 0$$

$$x(x^{\delta-1} e^{r-x} - 1) = 0$$

So an equilibrium exists at $x = 0$ and for the solution(s) of the equation $x^{\delta-1} e^{r-x} - 1 = 0$.

# Establishing $\bar{x}$

$$x^{\delta-1}e^{r-x} - 1 = 0$$

$$\Rightarrow 0 = x^{1-\delta} - e^{r-x} = g(x)$$

We know that

$$x \in [0, \infty).$$

$$g(0) = -e^{r}$$

$$\lim_{x\to\infty} g(x) = \infty$$

$\Leftrightarrow g(x) = 0$ has at least one solution on $(0, \infty)$ by the IVT.

# Establishing $\bar{x}$

We can now use $g'(x)$ to analyze the behavior of $g(x)$ on $[0, \infty)$.

$$g'(x) = (1 - \delta)x^{-\delta} + e^{r-x}$$

Since $\delta > 0, r > 0$,

$$g'(x) > 0, \forall x \in (0, \infty),$$

$$\Leftrightarrow g(x) \uparrow \text{on } (0, \infty),$$

$$\Leftrightarrow g(x) \text{ has exactly one solution } \bar{x}.$$

# Stability Analysis

So $f(x)$ has exactly two equilibria when $0 < \delta \leq 1$
Recall the following theorem

## Theorem

*An equilibrium $\hat{x}$ of the difference equation $x_{n+1} = f(x_n)$ :*
*(1) is Locally Asymptotically Stable (LAS) if $|f'(\hat{x})| < 1$.*
*(2) is Unstable if $|f'(\hat{x})| > 1$.*
*(3) if $|f'(\hat{x})| = 1$, stability is inconclusive.*

We can use these properties to analyze the stability at both equilibria $x = 0, \bar{x}$.

# Stability of Equilibria

$$f(x) = x^{\delta} e^{r-x}$$

$$f'(x) = x^{\delta-1} e^{r-x} (\delta - x)$$

Through analysis, we find that

$$|f'(0)| > 1 \Rightarrow 0 \text{ is unstable,}$$

And we find that

$$|f'(\bar{x})| < 1 \; \forall \; r < (\delta + 1) - (\delta - 1) \ln(\delta + 1)$$

# Case 1: $0 < \delta \leq 1$

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We will now observe the case when $\delta > 1$

# Case 2: $\delta > 1$

First we will look at some graphs of $f(x)$ with $\delta = 2$ and various values of $r$

Two positive equilibria

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One positive equilibrium

# Case 2: $\delta > 1$

First we will look at some graphs of $f(x)$ with $\delta = 2$ and various values of $r$

No Positive Equilibria

This leads to the following theorem:

**Theorem**

*Consider Ricker's Model where* $\delta > 1, r > 0.$ *The following statement is true:*
*(a) If* $r > (\delta - 1)(1 - \ln(\delta - 1))$ *it has three equilibria, the* 0 *equilibrium and two positive equilibria* $\tilde{x} < \delta - 1 < \bar{x}$.

# Proof

To prove this, we need to look at the equation

$$f(x) = x^\delta e^{r-x}.$$

Let us examine

$$x^\delta e^{r-x} = x.$$

$x = 0$ is always a fixed point for $f$, so for $x > 0$ we have:

$$\begin{aligned} 1 &= x^{\delta-1} e^{r-x} \\ 0 &= 1 - x^{\delta-1} e^{r-x} \end{aligned}$$

Now let $g(x) = 1 - x^{\delta-1} e^{r-x}$. The zeroes of $g(x)$ are the same as the positive fixed points of $f(x)$.

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From this graph of $g(x)$ with $\delta = 2$ and $r = 1.5$, we can observe that $g(x)$ has two positive equilibria in this case.
We have

$$g'(x) = x^{\delta-2}e^{r-x}(x - (\delta - 1)),$$

and $g'(x) = 0$ if $x = \delta - 1$, so the minimum of $g$ is less than 0 if $g(\delta - 1) < 0$. This is true if $r > (\delta - 1)(1 - \ln(\delta - 1))$ as stated, thus completes this proof.

### Theorem

*Consider the difference equation (1) where $\delta > 1$ and*

$$(\delta - 1)(1 - \ln(\delta - 1)) < r \leq (\delta + 1) - (\delta - 1)\ln(\delta + 1).$$

*Then the positive equilibrium $\bar{x} > \delta - 1$ is LAS.*

We demonstrate this theorem setting $\delta = 2$ and using varying values of $r$ and initial conditions $x_0$. Note that when $\delta = 2$, the theorem states that $\bar{x}$ is LAS for $1 < r \leq 3 - \ln 3 \approx 1.90$.

# Case 2: $\delta > 1$

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### Theorem

*Consider the difference equation (1) where $\delta > 1$ and*

$$(\delta - 1)(1 - \ln(\delta - 1)) < r \leq \delta - (\delta - 1)\ln(\delta).$$

*Let $\tilde{x}, \bar{x}$ ($\tilde{x} < \delta - 1 < \bar{x}$) be two positive equilibria of the same equation, and $\hat{x} \neq \tilde{x}$, satisfies $f(\hat{x}) = \tilde{x}$. Then the basin of attraction of the positive equilibrium $\bar{x}$ is the interval $(\tilde{x}, \hat{x})$.*

We will again demonstrate this theorem setting $\delta = 2$, $r = 1.2$, and using varying initial values $x_0$. For reference, $\tilde{x} \approx 0.493$, $\bar{x} \approx 1.77$, and $\hat{x} \approx 5.21$.

# Case 2: $\delta > 1$

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# Conclusion

By studying Ricker's model, we have found two scenarios.

1. When $\delta > 1$ for $x_0 < \tilde{x}$, the least positive equilibria, the sequence $\{x_n\}$ converges to zero.
2. When $\delta \leq 1$, two equlibria. $x = 0$ is an unstable equilibrium. The second equilibrium $x = \bar{x}$ is sometimes stable.

- Outlook
  - British Petroleum Oil Leak affecting the ecosystem in the Gulf of Mexico
  - The Chernobyl Nuclear Meltdown in 1986 still affecting wildlife in the Ukraine.

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# For Further Reading I


- Mark E. Burke
  Avoiding Extinction in a Managed Single Species Population Model by means of Anticipative Control
  *Dept. of Mathematics and Statistics, Limerick Ireland*.

- Leticia Aviles
  Cooperations and Non-Linear Dynamics: An Ecological Perspective on the Evolution of Sociality
  *Evolutionary Ecology Research*, (1999), 459–477.

- P.A. Stephens, W.J. Sutherland, R.P. Freckleton
  What is the Allee Effect?
  *Nordic Society Oikos*, (Oct. 1999), 185–190, Vol 87.