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THE EXISTENCE AND STABILITY OF EQUILIBRIA OF THE GENERALIZED RICKER'S MODEL

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ABSTRACT. To properly understand the behavior of the Generalized Ricker Population Model we must first do a complete stability analysis. This includes finding the equilibria of the model in certain general cases, determining stability at each point, and establishing attractivity of each of these equilibrium points.

Keywords: Allee Effect, Extinction, Basin of Attraction, Difference Equation, Attractor, Repellor, Equilibria, Local Asymptotical Stability, Global Asymptotical Stability.

1. INTRODUCTION

In this paper we study the existence and the stability of equilibria of the generalized Ricker's population model

$$x_{n+1} = x_n^{\delta} e^{r-x_n}, n = 0, 1, ... \tag{1.1}$$

where $r, \delta > 0$, the initial condition $x_0 > 0$. In this model x_n represents the population size (or density) in generation n. This model is intruduced in [2] and represents the generilization of classical Ricker's population model (case $\delta = 1$)

$$x_{n+1} = x_n e^{r-x_n}, n = 0, 1, ... \tag{1.2}$$

which has been thoroughly studied in the literature.

The Ricker Population Model, named after Bill Ricker in 1954, is a famous population model which expresses the expected number of individuals in a given generation as a function of the number of individuals in the previous generation. Since the beginnings of Ricker's Model, there have been numerous discoveries, mainly in the fishery sciences, but also in the biological sciences to study the dynamics of how a population will react to any given effect to their ecosystem. It is also known that the model exhibits "complex dynamics" including period doubling bifurcations and chaotic behavior.

For very small populations, the reproduction and survival rates of individuals increase with population density. This contrasts with larger

[1]LSU Summer Math Integrated Learning Experience
Date: Friday July 9, 2010.

populations, where greater population density slows the growth rate of the population due to competition.

2. Preliminaries

In this section we introduce some definitions and theorems that will be useful in the sequel.

Definition 2.1. $\bar{x}$ is an **equilibrium** of the equation

$$x_{n+1} = f(x_n), n = 0, 1, ... \tag{2.1}$$

if

$$\bar{x} = f(\bar{x}).$$

The corresponding solution $\{\bar{x}_n\}$ such that

$$\bar{x}_n = \bar{x}$$

is called also a **constant solution** or **steady-state solution**. Also, in such cases we say that $\bar{x}$ is a **fixed point** of the function f.

Definition 2.2. (Stability) (i) The equilibrium point $\bar{x}$ of Eq. (**??**) is called **(locally) stable** if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$|x_0 - \bar{x}| < \delta \quad \text{implies} \quad |x_n - \bar{x}| < \epsilon \quad \text{for} \quad n \geq 0.$$

Otherwise, the equilibrium $\bar{x}$ is called **unstable.**
(ii) The equilibrium point $\bar{x}$ of Eq. (**??**) is called **(locally) asymptotically stable** (**LAS**) if it is stable and there exists γ such that

$$|x_0 - \bar{x}| < \gamma \quad \text{implies} \quad \lim_{n\to\infty} |x_n - \bar{x}| = 0.$$

(iii) The equilibrium point $\bar{x}$ of Eq. (**??**) is called **globally asymptotically stable** (**GAS**)if for every x_0

$$\lim_{n\to\infty} |x_n - \bar{x}| = 0.$$

(iv) The equilibrium point $\bar{x}$ of Eq. (**??**) is called **globally asymptotically stable relative to a set** $S \subset \mathbb{R}$ if it is asymptotically stable, and if for every $x_0 \in S$,

$$\lim_{n\to\infty} |x_n - \bar{x}| = 0.$$

(v) The equilibrium point $\bar{x}$ of Eq. (**??**) is said to be an **attractor with the basin of attraction** $S \subset \mathbb{R}$ if

$$\lim_{n\to\infty} x_n = \bar{x}$$

for every solution with $x_0 \in S$.
(vi) The equilibrium point $\bar{x}$ of Eq. (**??**) is said to be a **repeller** if there exists $\eta > 0$, such that for all x_0 with $0 < |x_0 - \bar{x}| < \eta$, there exists an integer $N \geq 1$ such that

$$|x_N - \bar{x}| \geq \eta.$$

(vii) The equilibrium point $\bar{x}$ of Eq. (**??**) is called **(locally) semistable from the right (left)** if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$0 < x_0 - \bar{x} < \delta \; (-\delta < x_0 - \bar{x} < 0) \text{ implies } |x_n - \bar{x}| < \epsilon \text{ for } n \geq 0.$$

(viii) The equilibrium point $\bar{x}$ of Eq. (**??**) is called **(locally) asymptotically semistable from the right (left)** if it is (locally) semistable from the right (left) and there exists γ such that

$$0 \leq x_0 - \bar{x} < \gamma \quad (-\gamma < x_0 - \bar{x} \leq 0) \quad \text{implies} \quad \lim_{n\to\infty} |x_n - \bar{x}| = 0.$$

We have the following result:

Theorem 2.3. *Let $\bar{x}$ be an equilibrium of the difference equation (**??**) where f is a continuously differentiable function at $\bar{x}$.*
(i) If

$$|f'(\bar{x})| < 1$$

*then the equilibrium $\bar{x}$ is **locally asymptotically stable**.*
(ii) If

$$|f'(\bar{x})| > 1$$

*then the equilibrium $\bar{x}$ is **unstable**.*

Theorem 2.4. *Let $\bar{x}$ be an equilibrium of the difference equation (**??**) where f', f'', f''' are continuous at $\bar{x}$ such that $f'(\bar{x}) = 1$.*
(i) If

$$f''(\bar{x}) \neq 0$$

*then the equilibrium $\bar{x}$ is **semistable.** More precisley, if $f''(\bar{x}) > 0$, $\bar{x}$ is semistable from the left and if $f''(\bar{x}) < 0$, $\bar{x}$ is semistable from the right.*
(ii) If

$$f''(\bar{x}) = 0 \quad \text{and } f'''(\bar{x}) > 0$$

*then the equilibrium $\bar{x}$ is **unstable**.*
(iii) If

$$f''(\bar{x}) = 0 \quad \text{and } f'''(\bar{x}) < 0$$

*then the equilibrium $\bar{x}$ is **(locally) asymptotically stable**.*

Theorem 2.5. *Let $\bar{x}$ be an equilibrium of the difference equation (??) where f', f'', f''' are continuous at $\bar{x}$ such that $f'(\bar{x}) = -1$. Let $\mathbf{S}f$ be a Schwarzian derivative of f, defined by*

$$\mathbf{S}f(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left[\frac{f''(x)}{f'(x)}\right]^2$$

(i) If

$$\mathbf{S}f(\bar{x}) > 0$$

then the equilibrium $\bar{x}$ is ***unstable****.*
(ii) If

$$\mathbf{S}f(\bar{x}) < 0$$

then the equilibrium $\bar{x}$ is ***(locally) asymptotically stable****.*

3. Case $\delta > 1$

In this section we examine the existence and stability of equilibria of the equation (??) in the case $\delta > 1$. First, let us consider the function f defined by

$$f(x) = x^\delta e^{r-x} \tag{3.1}$$

for $x \geq 0$, and $\delta > 1, r > 0$. The follwing three graphs represent the function f in the case $\delta = 2$ for different values of r. Note that in all three cases 0 is always one of the equilibria. Equillibria are intersections of the graph of function f and the line $y = x$ (dashed line). In the case $\delta = 2, r = 1.5$ we see that there are two positive equilibria. In the case $\delta = 2, r = 1$ there is only one positive equilibrium. In the case $\delta = 2, r = 0.5$ there are no positive equilibria, only a unique equilibrium 0. These three cases are typical for this model.

Two positive equilibria

One positive equilibrium

No positive equilbria.

Theorem 3.1. *Consider the difference equation (??) where $\delta > 1, r > 0$. The following statements are true:*

(a) If $r < (\delta - 1)(1 - \ln(\delta - 1))$ the equation (??) has one equilibrium 0.

(b) If $r = (\delta - 1)(1 - \ln(\delta - 1))$ the equation (??) has two equilibria 0 *and one positive equilibruim $\bar{x} = \delta - 1$.*

*(c) If $r > (\delta - 1)(1 - \ln(\delta - 1))$ the equation (**??**) has three equilibria 0 and two positive equilibria $\bar{x}$ and $\tilde{x}$, such that $0 < \tilde{x} < \delta - 1 < \bar{x}$.*

Proof. To prove this, we need to look at the equation $x^\delta e^{r-x} = x$.

$$x = x^\delta e^{r-x}$$

$x = 0$ is always a solution, so for $x > 0$ we have:

$$\begin{aligned} 1 &= x^{\delta-1} e^{r-x} \\ 0 &= 1 - x^{\delta-1} e^{r-x} \end{aligned}$$

Now, let

$$f(x) = x^\delta e^{r-x}$$

and

$$g(x) = 1 - x^{\delta-1} e^{r-x} \tag{3.2}$$

The zeroes of $g(x)$ are the same as the fixed points of $f(x)$.

$$\begin{aligned} g'(x) &= x^{\delta-2} e^{r-x}(x - (\delta - 1)) \\ g'(x) &= 0 \text{ if } x = \delta - 1 \end{aligned}$$

Since $g''(\delta - 1) = (\delta - 1)^{\delta-2} e^{r-(\delta-1)} > 0$ for $\delta > 1$, $\delta - 1$ is a minimum of g. For $x \in (0, \delta - 1)$, $x - (\delta - 1) < 0$, so $g(x)$ is decreasing on $(0, \delta - 1)$. Similarly, $x - (\delta - 1) > 0$ for $x \in (\delta - 1, \infty)$, so $g(x)$ is increasing on $(\delta - 1, \infty)$. Thus, $\delta - 1$ is the global minimum of g. Observe that $g(0) = 1$ and $\lim_{x\to\infty} g(x) = 1$. If $g(\delta - 1) > 0$ then $g(x) > 0$ for all values of x, so g has no zeroes. This is true when

$$\begin{aligned} 1 - (\delta - 1)^{\delta-1} e^{r-(\delta-1)} &> 0 \\ 1 &> (\delta - 1)^{\delta-1} e^{r-(\delta-1)} \\ (\delta - 1)^{-(\delta-1)} &> e^{r-(\delta-1)} \\ -(\delta - 1)\ln(\delta - 1) &> r - (\delta - 1) \\ r &< (\delta - 1)(1 - \ln(\delta - 1)) \end{aligned}$$

so when $r < (\delta - 1)(1 - \ln(\delta - 1))$, $g(x) > 0$ for all x and the proof of part (a) is complete.

A similar argument shows $g(\delta - 1) \leq 0$ when $r \geq (\delta - 1)(1 - \ln(\delta - 1))$, with equality when $r = (\delta - 1)(1 - \ln(\delta - 1))$. Thus, when $r = (\delta - 1)(1 - \ln(\delta - 1))$, $f(x)$ has exactly one nonzero equilibrium (at $\bar{x} = \delta - 1$)), when $r > (\delta - 1)(1 - \ln(\delta - 1))$ $f(x)$ has two nonzero equilibria, $\tilde{x} < \delta - 1$, and when $r < (\delta - 1)(1 - \ln(\delta - 1))$. □

It turns out that when $\delta = 2$, $(\delta - 1)(1 - \ln(\delta - 1)) = 1$, so in the case $\delta = 2$, if $r < 1$, $f(x)$ has no positive equilibria, if $r = 1$ $f(x)$ has one positive equilibrium, and for $r > 1$ there are two positive equilibria.

Next we will examine the stability of equilibria.

After describing the equilibria of $f(x)$, the next step is to tell when those equilibria are locally asymptotically stable. From the graph, it seems that 0 and $\bar{x}$ are attracting, and $\tilde{x}$ is repelling.

Theorem 3.2. *Consider the difference equation (**??**) where $\delta > 1, r > 0$. Then the equilibrium at 0 is locally asymptotically stable.*

Proof. As a direct result of Theorem **??**. Since $f'(x) = x^{\delta-1}e^{r-x}(\delta - x)$ we have $|f'(0)| = 0 < 1$ so 0 is locally asymptotically stable. □

Theorem 3.3. *Consider the difference equation (**??**) where $\delta > 1$ and $r = (\delta - 1)(1 - \ln(\delta - 1))$. Then the positive equilibrium $\bar{x} = \delta - 1$ is semistable from the right.*

Proof. Since $f'(\bar{x}) = 1$ and

$$\begin{aligned}
f''(\bar{x}) &= f''(\delta - 1) \\
&= (\delta - 1)^{\delta-2}e^{((\delta-1)-(\delta-1)\ln(\delta-1))-(\delta-1)}((\delta - 1)^2 - \delta(\delta - 1)) \\
&= (\delta - 1)^{\delta-2}e^{-(\delta-1)\ln(\delta-1)}((\delta - 1)^2 - \delta(\delta - 1)) \\
&= (\delta - 1)^{\delta-2}(\delta - 1)^{-(\delta-1)}((\delta - 1)^2 - \delta(\delta - 1)) \\
&= (\delta - 1)^{(\delta-2)-(\delta-1)}((\delta - 1)^2 - \delta(\delta - 1)) \\
&= (\delta - 1)^{-1}((\delta - 1)^2 - \delta(\delta - 1)) \\
&= (\delta - 1 - \delta) \\
&= -1 < 0,
\end{aligned}$$

$\bar{x} = \delta - 1$ is semistable from the right by Theorem **??**. □

Theorem 3.4. *Consider the difference equation (**??**) where $\delta > 1$ and*

$$(\delta - 1)(1 - \ln(\delta - 1)) < r \leq (\delta + 1) - (\delta - 1)\ln(\delta + 1).$$

Then the positive equilibrium $\bar{x} > \delta - 1$ is LAS.

Proof. $\bar{x}$ is LAS if $|f'(\bar{x})| < 1$ by Theorem **??**. Since $\bar{x}$ is an equilibrium of $f(x)$, $\bar{x}$ is LAS if $|\delta - \bar{x}| < 1$. This is equivalent to $\delta - 1 < \bar{x} < \delta + 1$. Since $\bar{x} > \delta - 1$, we must show when $\bar{x} < \delta + 1$. As $g(x) = 1 - x^{\delta-1}e^{r-x}$ is increasing on $(\delta-1,\infty)$, $\bar{x}<\delta+1$ is equivalent to $g(\bar{x})=0<g(\delta+1)$.

$$\begin{aligned}
0 &< g(\delta+1)\\
0 &< 1-(\delta+1)^{\delta-1}e^{r-(\delta+1)}\\
(\delta+1)^{\delta-1}e^{r-(\delta+1)} &< 1\\
e^{r-(\delta+1)} &< (\delta+1)^{-(\delta-1)}\\
r-(\delta+1) &< -(\delta-1)\ln(\delta+1)\\
r &< (\delta+1)-(\delta-1)\ln(\delta+1)
\end{aligned}$$

It was proved in Theorem **??** that the difference equation (**??**) has equilibria $\bar{x}>\delta-1$ if and only if $r>(\delta-1)(1-\ln(\delta-1))$, so for $(\delta-1)(1-\ln(\delta-1))<r<(\delta+1)-(\delta-1)\ln(\delta+1)$, $\bar{x}$ is LAS.

A similar argument shows that if $\bar{x}=\delta+1$, then $r=(\delta+1)-(\delta-1)\ln(\delta+1)$ and $|f'(\bar{x})|=|-1|\not<1$. However, by Theorem **??** $\bar{x}$ is LAS if $\mathbf{S}f(\bar{x})<0$.

$$\mathbf{S}f(\bar{x})=\mathbf{S}f(\delta+1)=\frac{-3\delta+1}{2(\delta+1)}<0$$

since $\delta>1$ □

Theorem 3.5. *Consider the difference equation (**??**) where $\delta>1$ and*

$$r>(\delta-1)(1-\ln(\delta-1)).$$

Then the positive equilibrium $\tilde{x}<\delta-1$ is unstable.

Proof. Since $f'(\tilde{x})=\tilde{x}^{\delta-1}e^{r-\tilde{x}}(\delta-\tilde{x})$ and $\tilde{x}^{\delta-1}e^{r-\tilde{x}}=1$ we have $f'(\tilde{x})=\delta-\tilde{x}>1$ and $\tilde{x}$ is unstable. □

Next we study the attractivity of the equilibria. The following two technical lemmas will be useful.

Lemma 3.6. *$f(x)=x^{\delta}e^{r-x}$ is increasing on $(0,\delta)$ and decreasing on (δ,∞)*

Proof.

$$\begin{aligned}
f'(x) &= \delta x^{\delta-1}e^{r-x}-x^{\delta}e^{r-x}\\
&= x^{\delta-1}e^{r-x}(\delta-x)
\end{aligned}$$

For $x\in(0,\delta)$, $f'(x)>0$ since $x^{\delta-1}e^{r-x}>0$ for all $x>0$, and $\delta-x>0$ since $x<\delta$. For $x\in(\delta,\infty)$, $f'(x)<0$ since $x^{\delta-1}e^{r-x}>0$ for all $x>0$, and $\delta-x<0$ since $x>\delta$. Thus, $f(x)$ is increasing on $(0,\delta)$ and decreasing on (δ,∞) □

Lemma **??** is very helpful because, if $a,b\in(0,\delta)$, $a<b$ is equivalent to $f(a)<f(b)$. Similarly, for $a,b\in(\delta,\infty)$ $a<b$ if and only if $f(a)>f(b)$.

Lemma 3.7. *Consider the difference equation (**??**) where $\delta > 1$ and $r > (\delta-1)(1-\ln(\delta-1))$. Let $\tilde{x}, \bar{x}$ ($\tilde{x} < \delta - 1 < \bar{x}$) be two positive equilibria of the same equation. Then there exists $\hat{x} \neq \tilde{x}$, such that $f(\hat{x}) = \tilde{x}$.*

Proof. $f(x) = x^{\delta}e^{r-x}$ has a maximum at $x = \delta$ with $f(\delta) > \tilde{x}$, is decreasing on (δ, ∞) by Lemma **??**, and $\lim_{x\to\infty} f(x) = 0$, so by the Intermediate Value Theorem there exists some $\hat{x}$ in (δ, ∞) with $f(\hat{x}) = \tilde{x}$. □

Theorem 3.8. *Consider the difference equation (**??**) where $\delta > 1$ and $r > (\delta-1)(1-\ln(\delta-1))$. Let $\tilde{x}, \bar{x}$ ($\tilde{x} < \delta - 1 < \bar{x}$) be two positive equilibria of the same equation, and $\hat{x} \neq \tilde{x}$, satisfies $f(\hat{x}) = \tilde{x}$. Then the basin of attraction of the equilibrium at 0 is the set $[0, \tilde{x}) \cup (\hat{x}, \infty)$.*

Proof. Let $x_0 \in [0, \tilde{x})$. First we prove by induction that $\{x_n\}$ is contained within $[0, \tilde{x})$. The case $n = 0$ is true, since it is given that $x_0 \in [0, \tilde{x})$. Assume it is true for $n = k$, that is, $0 \leq x_k < \tilde{x}$. Since $f(x)$ is increasing on $(0, \delta)$ by Lemma **??**, and $f(0) = 0$, $f(x_k) = x_{k+1}$, $f(\tilde{x}) = \tilde{x}$ we have $0 \leq x_{k+1} < \tilde{x}$, proving that $\{x_n\} \in [0, \tilde{x})$ for all integers n. Now, using the observation that $f(x) < x$ on $(0, \tilde{x})$, we have $x_{n+1} = f(x_n) < x_n$, and $\{x_n\}$ is decreasing.

Since $\{x_n\}$ is both bounded and decreasing, it converges. Therefore,

$$\lim_{n\to\infty} x_n = x \in [0, \tilde{x}),$$

and

$$\begin{aligned} \lim_{n\to\infty} x_{n+1} &= \lim_{n\to\infty} x_n^{\delta} e^{r-x_n} \\ x &= x^{\delta} e^{r-x} \end{aligned}$$

Since the only solution to $x = x^{\delta}e^{r-x}$ in $[0, \tilde{x})$ is $x = 0$, $\{x_n\}$ converges to 0 if $x_0 \in [0, \tilde{x})$.

For $\hat{x} < x_0 < \infty$, and since $f(x)$ is decreasing on $(\hat{x}, \infty)$, we have $\lim_{x\to\infty} f(x) = 0 < f(x_0) < f(\hat{x}) = \tilde{x}$. This is simply have $0 < x_1 < \tilde{x}$ which was previously shown to converge to 0. □

Theorem 3.9. *Consider the difference equation (**??**) where $\delta > 1$ and*

$$(\delta-1)(1-\ln(\delta-1)) < r \leq \delta - (\delta-1)\ln(\delta).$$

Let $\tilde{x}, \bar{x}$ ($\tilde{x} < \delta - 1 < \bar{x}$) be two positive equilibria of the same equation, and $\hat{x} \neq \tilde{x}$, satisfies $f(\hat{x}) = \tilde{x}$. Then the basin of attraction of the positive equilibrium $\bar{x}$ is the interval $(\tilde{x}, \hat{x})$.

Proof. First we will show that the condition $r \leq \delta - (\delta-1)\ln(\delta)$ is equivalent to $\bar{x} \leq \delta$. Following the same arguments as in the proof of

Theorem 2.3 the condition $\bar{x} \leq \delta$ is equivalent to $0 = g(\bar{x}) \leq g(\delta)$, where $g(x) = 1 - x^{\delta-1}e^{r-x}$. So we obtain

$$1 - \delta^{\delta-1}e^{r-\delta} \geq 0$$

which is equivalent to $r \leq \delta - (\delta - 1)\ln(\delta)$.

The following cases are possible:

Case 1: $\tilde{x} < x_0 \leq \bar{x}$. First we show that $\{x_n\}$ converges to $\bar{x}$ if $x_0 \in (\tilde{x}, \bar{x}]$. The argument is very similar to the proof of Theorem **??**. Let $x_0 \in (\tilde{x}, \bar{x}]$. We prove, by induction, that $\{x_n\}$ is contained within $(\tilde{x}, \bar{x}]$. The case $n = 0$ is true, since it is given that $x_0 \in (\tilde{x}, \bar{x}]$. Assume the case $n = k$ for some positive integer k is true, $\tilde{x} < x_k \leq \bar{x}$. Since $f(x)$ is increasing on $(0, \delta)$ by Lemma **??**, $f(\tilde{x}) < f(x_k) \leq f(\bar{x})$, but, $f(\tilde{x}) = \tilde{x}$, $f(x_k) = x_{k+1}$, and $f(\bar{x}) = \bar{x}$ so using the assumption that $\tilde{x} < x_k \leq \bar{x}$ we have $\tilde{x} < x_{k+1} \leq \bar{x}$, proving that $\{x_n\} \in (\tilde{x}, \bar{x}]$ for all integers n. Now, using the observation that $f(x) > x$ on $(\tilde{x}, \bar{x}]$, we have $x_{n+1} = f(x_n) > x_n$, and $\{x_n\}$ is increasing. Since $\{x_n\}$ is both bounded and increasing, it converges. Therefore,

$$\lim_{n\to\infty} x_n = x \in (\tilde{x}, \bar{x}],$$

and

$$\begin{aligned} \lim_{n\to\infty} x_{n+1} &= \lim_{n\to\infty} x_n^{\delta} e^{r-x_n} \\ x &= x^{\delta} e^{r-x} \end{aligned}$$

Since the only solution to $x = x^{\delta}e^{r-x}$ in $(\tilde{x}, \bar{x}]$ is $x = \bar{x}$, $\{x_n\}$ converges to $\bar{x}$ if $x_0 \in (\tilde{x}, \bar{x}]$.

Case 2. $\bar{x} < x_0 \leq \delta$. The proof is similar to the previous one and is omitted.

Case 3. $\delta < x_0 \leq x^{\star}$ with $f(x^{\star}) = \bar{x}$.

$$\begin{aligned} \delta < x_0 &\leq x^{\star} \\ f(\delta) > f(x_0) &\geq f(x^{\star}) \\ \delta > f(\delta) > x_1 &\geq \bar{x} \end{aligned}$$

and this becomes case 2.

Case 4. $x^{\star} < x_0 < \hat{x}$.

$$\begin{aligned} x^{\star} < x_0 &< \hat{x} \\ f(x^{\star}) > f(x_0) &> f(\hat{x}) \\ \tilde{x} < x_1 &< \bar{x} \end{aligned}$$

so we obtain case 1. $\square$

4. Case $0 < \delta \leq 1$

Theorem 4.1. *Consider the difference equation (**??**) where $0 < \delta \leq 1, r > 0$. The equation has two equilibria, 0 and a positive equilibrium $\bar{x}$.*

Proof.

$$\begin{aligned} x &= f(x) \\ x &= x^{\delta} e^{r-x} \\ 0 &= x - x^{\delta} e^{r-x} \\ 0 &= x(1 - x^{\delta} e^{r-x}) \end{aligned}$$

So the solutions to $f(x) = x$ are $x = 0$ and the solutions to $1 - x^{\delta-1} e^{r-x} = 0$

Since $x \neq 0$

$$\begin{aligned} 0 &= 1 - x^{\delta-1} e^{r-x} \\ 0 &= x^{1-\delta} - e^{r-x} \end{aligned}$$

Now let $h(x) = x^{1-\delta} - e^{r-x}$. $h(0) = -e^r < 0$ and $\lim_{x\to\infty} h(x) = \infty$. By the Intermediate Value Theorem, since $h(x)$ is a real valued, continuous function, $h(x)$ must pass through the line $x = 0$ on the interval $(0, \infty)$ at least once. So Equation (**??**) has at least one positive equilibrium. Observe that $h'(x) = (1-\delta)x^{-\delta} + e^{r-x}$. Since $0 < \delta < 1$, $h'(x) > 0$ for all $x \in (0, \infty)$ and $h(x)$ is strictly increasing on $(0, \infty)$. Thus (**??**) has exactly one positive equilibrium $\bar{x} \in (0, \infty)$. □

Theorem 4.2. *Consider the difference equation (**??**) where*

$$0 < \delta \leq 1, r > 0.$$

The following statements are true:

(a) 0 equilibrium is unstable.

(b) If $r < (\delta+1) - (\delta-1)\ln(\delta+1))$, then the positive equilibrium $\bar{x}$ is locally asymptotically stable.

Proof. Part (a): By Theorem **??**, if $|f'(0)| > 1$ then $x = 0$ is unstable.

$$f'(x) = x^{\delta-1} e^{r-x}(\delta - x) \tag{4.1}$$

For $0 < \delta < 1$, $f'(0)$ is undefined, since $x^{\delta-1} = 1/x^{1-\delta}$. However, $\lim_{x\to 0+} |f'(x)| = \infty > 1$ so the 0 equilibrium is unstable.

When $\delta = 1$, $f'(0) = e^r > 1$ since $r > 0$ so the 0 equilibrium is again unstable.

Part (b): By Theorem **??**, if $|f'(\bar{x})| < 1$ then $\bar{x}$ is LAS. Since $f'(\bar{x}) = \bar{x}^{\delta-1}e^{r-\bar{x}}(\delta - \bar{x})$. and $h(\bar{x}) = 0$, $f'(\bar{x}) = \delta - \bar{x}$.

$$\begin{aligned} |f'(\bar{x})| &< 1 \\ -1 &< \delta - \bar{x} < 1 \\ \delta - 1 &< \bar{x} < \delta + 1 \end{aligned}$$

As $\bar{x} \in (0, \infty)$, $\bar{x} > \delta - 1$, so when $\bar{x} < \delta + 1$, $\bar{x}$ is LAS. Since $h(x)$ is strictly increasing on $(0, \infty)$, $\bar{x} < \delta + 1$ is equivalent to $h(\bar{x}) = 0 < h(\delta + 1)$.

$$\begin{aligned} h(\delta + 1) &> 0 \\ (\delta + 1)^{1-\delta} - e^{r-(\delta+1)} &> 0 \\ e^{r-(\delta+1)} &< (\delta + 1)^{1-\delta} \\ r - (\delta + 1) &< ln(\delta + 1)^{1-\delta} \\ r &< (\delta + 1) + (1 - \delta)ln(\delta + 1) \end{aligned}$$

So when $r < (\delta + 1) + (1 - \delta)ln(\delta + 1)$, $\bar{x}$ is LAS. □

Theorem 4.3. *Consider the difference equation (**??**) where $\delta \geq 1$ and $r \leq (\delta) + (1 - \delta)ln(\delta)$. Then the basin of attraction for $\bar{x}$ is $(0, \infty)$.*

Proof. This proof is very similar to the proof for Theorem **??** and is omitted. □

5. CONCLUSION

The limiting factor in our understanding of the Generalized Ricker's Population Model is the basin of attraction for $(\delta) - (\delta - 1)\ln(\delta) < r < (\delta + 1) - (\delta - 1)\ln(\delta + 1)$. We were unable to prove this case, but a computer model

shows that the following conjecture is true.

Conjecture 5.1. *Consider the difference equation (**??**) where $\delta > 0$ and*

$$(\delta) - (\delta - 1)\ln(\delta) < r < (\delta + 1) - (\delta - 1)\ln(\delta + 1).$$

*Then the basins of attraction for the positive LAS equilibrium $\bar{x}$ are the same as the basins of attraction specified in Theorems **??** and **??***

5.1. **Extrapolation.** The Ricker's population model, discovered in fishery science, is used for numerous fields, especially in the biological sciences. One such application this model can be used in is a certain disaster that is occuring right now close to home, the British Petroleum Oil leak. By applying Ricker's model, we can make a rough approximation on how the ecosystems will be affected by the disaster. To do this we need to consider the non autonomous generalized Ricker's Model:

$$x_{n+1} = x_n^{\delta} e^{r_n - x_n}$$

where $\{r_n\}$ is a given positive sequence. This reflects the case where the environment changes, that is, the carrying capacity is not constant. Since the oil leak on April 20, 2010, the oil that has leaked from the broken well has killed much of the microorganisms that thrive on oxygen and sunlight. When the microorganisms can't receive their oxygen because of the oil, they are starved and die off. When the microorganisms die off, it creates a disruption in the food chain. The larger organisms that thrive off the smaller ones can't eat and die off, thus creating a "domino effect". According to the definition of the carrying capacity, once the food is gone, the remaining species of the ecosystem have no choice but to either die off, or migrate to another location in order to survive.

We are able to observe that this conjecture is true with the help of a computer, but are unable to prove it at this time.

6. Acknowledgements

We would like to thank Dr. Vlajko L. Kocic for inspiring our interest in Mathematical Modeling in the Life Sciences and for all the help he provided both in class and with our project.

We would also like to thank Robert Viator our mentor for providing much needed assistance in the project and Beamer presentation.

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