The Gamma Function

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SMILE REU

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Motivation and History

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- *"Each generation has found something of interest to say about the gamma function. Perhaps the next generation will also."*
  -Philip J. Davis

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The Gamma function is analytic for all complex numbers except the non-positive integers. The function has simple poles at these values, with residues given by $\frac{(-1)^s}{s!}$.

Alternate Definitions and Functional Equations

A few of the most useful ones:

1. $\Gamma(s) = \lim_{n\to\infty} \frac{n^s n!}{s(s+1)...(s+n)}$ for $s \neq 0, -1, -2, ...$
2. $\frac{1}{\Gamma(s)} = se^{\gamma s}\prod_{n=1}^{\infty}(1+\frac{s}{n})e^{-s/n} \ \forall\ s.$
3. $\Gamma(s+1) = s\Gamma(s).$
4. $\Gamma(s)\Gamma(1-s) = \frac{\pi}{sin \pi s}.$

The Digamma and Polygamma Functions

- The Digamma function, $\Psi^{(0)}(x)$ is defined as the derivative of the logarithm of $\Gamma(x)$.

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- The Polygamma function, $\Psi^{(k)}(x)$ is the generalization to higher derivatives.

$$\Psi^{(k)}(x) = \frac{d^{k+1}}{dx^{k+1}}(\log \Gamma(x))$$

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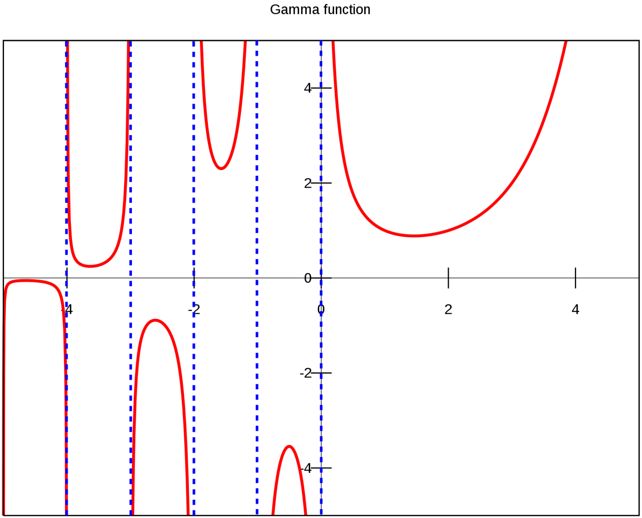
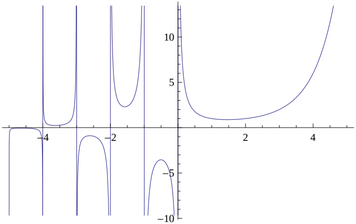
- $\gamma = \lim_{r \to \infty}(\log r - 1 - \frac{1}{2} - \frac{1}{3} - \ldots - \frac{1}{r})$
- It is unknown whether γ is algebraic or transcendental.

Graph of the Gamma Function



Overview of Behavior

We looked at several features of the graph:



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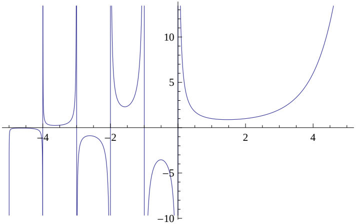

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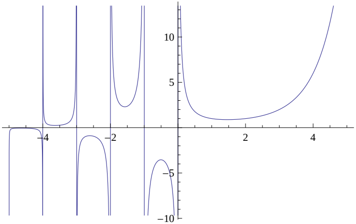

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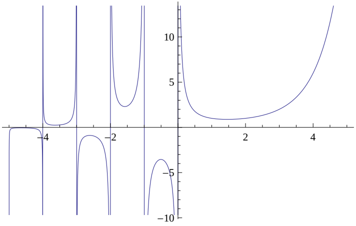

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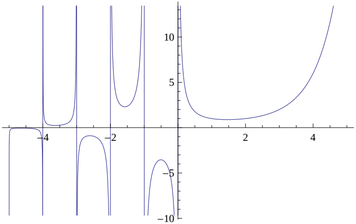
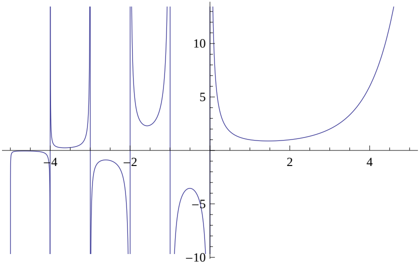


- The area under the curve.
- Critical points of the graph for negative values shift progressively leftwards.
- The graph restricted to intervals between the discontinuities looks like a squeezed segment of the graph in the positive regime.
- Critical points for negative values approach zero.

Questions About the Integral of $\Gamma(x)$

When considering the graph of the Gamma Function, one might be lead to consider several things.



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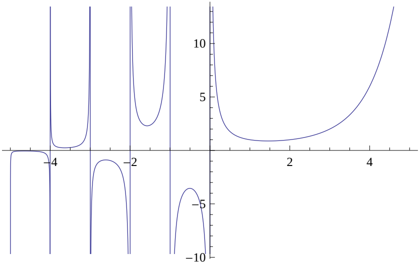

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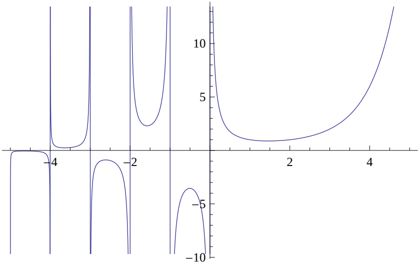

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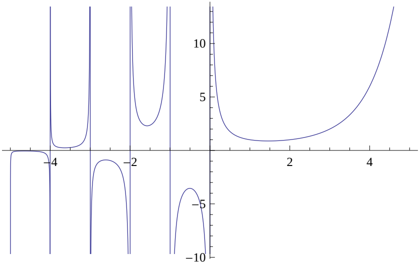

- Does the integral of $\Gamma(x)$ converge if one of the bounds of integration is a point of discontinuity?
- Does the integral of $\Gamma(x)$ converge if we integrate over a point of discontinuity?
- Does the integral of $\Gamma(x)$ from $-\infty$ to any real number converge?

Behavior Near the Points of Discontinuity

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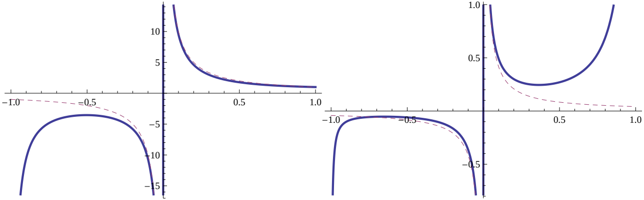
$$\Gamma(x-k) \approx \frac{1}{x(k!)}$$



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- $\int_{0}^{\epsilon} \Gamma(x-k)dx \approx \frac{1}{k!} \int_{0}^{\epsilon} \frac{dx}{x} \to \pm\infty$
- $\left| \int_{a}^{b} \Gamma(x)dx \right| < \infty$ for all a,b that are neither negative integers nor 0.

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$$\Rightarrow \sum_{k=0}^{\infty} \left| \int_{-\frac{1}{2}}^{\frac{1}{2}} \Gamma(x-k)dx \right| \leq \left| \int_{-\frac{1}{2}}^{\frac{1}{2}} \Gamma(x-k)dx \right| \sum_{k=0}^{\infty} \frac{2^k}{(2k-1)!!}$$

Critical Points

On the n^{th} interval $(-n, -n+1)$, let x^* be the x-coordinate of the critical point of the gamma function on this interval.
Let:

$$x_n = x^* + n \text{ for } 0 < x_n < 1$$

$$\Psi(x) = \sum_{k=1}^{n} \frac{1}{x-k}$$

Claim: x_n is unique

$$\lim_{n \to \infty} x_n \log n = 1$$

Value of the Gamma Function at Critical Points

Let d_n denote the value of the gamma function at its critical point on the n^{th} interval $(-n, -n+1)$. We have:

$$|d_n| = \left|\frac{\Gamma(x_n)}{\prod_{k=1}^{n}(x_n - k)}\right|$$

Let:

$$B = \left|\frac{\prod_{k=1}^{n}(x_n - k)}{n!}\right|$$

Since $\lim_{n\to\infty} x_n \log n = 1$, we have :

$$\lim_{n\to\infty} B = \frac{1}{e}$$

Using this gives us:

$$\lim_{n\to\infty} \frac{n!|d_n|}{\log n} = e$$

The Bluntness of The Gamma Function

The gamma function seems to flatten out when we move toward the negative side. To explain this, we will try to analyze the solution to the following equation:

$$\Gamma'(x) = \alpha$$

where α is an arbitrary positive real number.
On the n^{th} interval, let the solution on this interval be: $x_n^* - n$. This means $0 < x_n^* < 1$ and:

$$\Gamma'(x_n^* - n) = \alpha$$

The Bluntness of the Gamma Function

We start by proving the following limit for any real s, where $0 < s < 1$:

$$\lim_{n \to \infty} \Gamma'(s - n) = 0$$

This means:

$$\lim_{n \to \infty} x_n^* = 1$$

The Bluntness of the Gamma Function

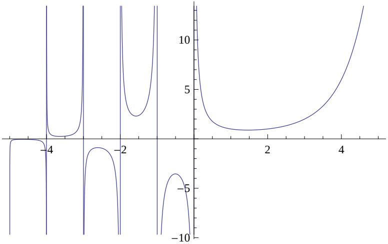
Next we will analyze how fast the sequence $\{x_n^*\}$ goes to 1 by analyzing how fast the sequence $\{y_n\}$ goes to zero, where $y_n = 1 - x_n^*$.

$$\lim_{n\to\infty} y_n^2(n-1)! = \frac{1}{\alpha}$$

Now we can use this limit to go back and prove that when n is large enough, the sequence $\{x_n^*\}$ is strictly moving to the right.

Conclusion

- The integral of the Gamma function from $-\infty$ to any finite, positive number converges.
- The critical points of the function on the negative real line migrate towards the asymptotes.
- The Gamma function flattens out as we move to more negative values.



Bibliography

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