GAMMA FUNCTION

ABSTRACT. In this paper we explore the history and properties of the Gamma function in an analytic number theoretical context. We analyze the behavior of the Gamma function at its critical points and points of discontinuity, and discuss the convergence of the integral.

1. INTRODUCTION

1.1. **History and Motivation.** In the early 16th century, Leonhard Euler and others attempted to expand the domain of the factorial to the real numbers. This cannot be done with elementary functions, however, with the notions of limits and integrals from the calculus, there were a few expressions developed. In 1738, Euler published the following forms of the generalized factorial:

- $n! = \prod_{k=1}^{\infty} \frac{(1+1/k)^n}{1+n/k}$
- $n! = \int_0^1 (-\log s)^n ds$ for $n > 0$

For the time being, we will refer to these as the product and integral definitions.

In the 19th century, Carl Friedrich Gauss rewrote Euler's product definition to extend the domain to the complex plane, rather than simply the real numbers. This expression uses a limiting process on a series of intermediate functions to represent the factorial.

$$\Gamma_r(x) = \frac{r!r^x}{x(1+x)(2+x)\dots(r+x)}$$

$$\Gamma(x) = \lim_{r\to\infty} \Gamma_r(x)$$

This was also the time period that the notation $\Gamma(x)$ was applied to the concept. It was named in 1811 by Adrien-Marie Legendre, who also transformed the integral definition in a very simple way to extend its domain:

$$n! = \int_0^1 (-\log s)^n ds$$

$$\text{Let } t = -\log s.$$

$$s = -e^t$$

$$dt = -\frac{1}{s}ds = e^t ds$$
$$\Gamma(n) = \int_0^\infty t^{n-1}e^{-t}dt,$$
which is the more common integral definition that we see today.

A few years later, Karl Weierstrass reworked Gauss' expression in the following manner:

$$\Gamma_r(x) = \frac{e^{x\log r}}{x(1+x/1)(1+x/2)\ldots(1+x/r)}$$
$$e^{x\log r} = e^{\gamma x}e^{x+x/2+x/3+\ldots+x/r}$$
$$\Gamma_r(x) = e^{\gamma x}\frac{1}{x}\frac{e^x}{(1+x/1)}\frac{e^{x/2}}{(1+x/2)}\cdots\frac{e^{x/r}}{(1+x/r)}$$
$$\frac{1}{\Gamma(x)} = \lim_{r\to\infty}\frac{1}{\Gamma_r(x)} = xe^{-\gamma x}\prod_{r=1}^{\infty}(1+\frac{x}{r})e^{-x/r},$$
where $\gamma = \lim_{r\to\infty}(\log r - 1 - \frac{1}{2} - \frac{1}{3} - \ldots - \frac{1}{r})$ is Euler's gamma constant.

This formulation is defined in terms of zeroes rather than poles. This actually inspired Weierstrass to prove the Weierstrass Factorization Theorem, which says that any entire function can be written as a product of its zeroes over $\mathbb{C}$.

In 1922, Bohr and Mollerup confronted the issue of whether these expressions of the gamma function were equivalent, and, relatedly, whether the Gamma function as defined is unique. Their results are summarized in the following theorem:

Theorem 1.1 (Bohr-Mollerup Theorem)**.** *If $\Gamma_1 = \Gamma_2$ on the integers and $\Gamma_1(x) = \Gamma_2(x)$ is logarithmically convex for $Re(x) > 0$, then $\Gamma_1 = \Gamma_2 \ \forall \ x \in \mathbb{R}$.*

1.2. **Log convexity.** The previous theorem guarantees uniqueness for the Gamma function if it is logarithmically convex. The following theorem and its proof can be found in Emil Artin's paper, *The Gamma Function.*

Definition 1.2. $f(x)$ is said to be logarithmically convex if the function $\log f(x)$ is convex.

Theorem 1.3. *Let $f(x)$ be a twice differentiable function. Then $f(x)$ is convex if and only if $f''(x) \geq 0$ for all x in the domain.*

Now we will show that the Gamma function satisfies the above criterion for log convexity.

Proposition 1.4. $\Gamma(x)$ is logarithmically convex for all $x > 0$.

Proof:

$$\frac{d}{dx}(\log\Gamma(x)) = \frac{\Gamma'(x)}{\Gamma(x)}$$

$$= -\frac{1}{x} + \gamma = \sum_{r=1}^{\infty}(\frac{1}{r+x} - \frac{1}{r})$$

$$\frac{d^2}{dx^2}(\log\Gamma(x)) = \frac{1}{x^2} + \sum_{i=1}^{\infty}\frac{1}{(x+i)^2}$$

It is clear that this expression is strictly positive, so the Gamma function is log convex. □

This establishes the uniqueness of the Gamma function as the extension of the factorial.

1.3. **Properties.** The Gamma function satisfies the following functional equations:

- $\Gamma(z+1) = z\Gamma(z)$
- $\Gamma(1) = 1$
- $\Gamma(n) = (n-1)!$ for $n \in \mathbb{N}$
- $\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin \pi z}$

We express the derivatives of the Gamma function in terms of another special function, the Polygamma function, $\Psi^{(k)}$.

$$\Psi^{(m)}(z) = (-1)^{(m+1)}\int_0^{\infty}\frac{t^m e^{-zt}}{1-e^{-t}}dt$$

More important in our context is the following form:

$$\Psi^{(m)}(z) = (\frac{d}{dz})^{m+1}(\log\Gamma(z)).$$

The digamma is the first polygamma function, in the case $m = 0$.

$$\Psi^{(0)}(z) = \frac{\Gamma'(z)}{\Gamma(z)}$$

$$\Gamma'(z) = \Psi^{(0)}(z)\Gamma(z)$$

This leads us to use the following expression for the derivatives of gamma:

$$\Gamma^{(k)}(z) = \int_0^{\infty} t^{z-1}e^{-t}(\log t)^n dt.$$

The Gamma function is analytic for all values $z \in \mathbb{C}$ except for the non-positive integers. At these points the function has simple poles. The residue for the point at $n = 0, -1, -2, \ldots$ is $\frac{(-1)^{-n}}{(-n)!}$.

2. Notation

Here is some notation that we will use later in this paper:

For x, s, and n real numbers, we define:

$$\begin{aligned}
\Psi(x) &:= \frac{\Gamma'(x)}{\Gamma(x)} \\
E^{+}(x) &:= \left(1 + \frac{1}{x}\right)^{x} \\
E^{-}(x) &:= \left(1 - \frac{1}{x}\right)^{-x} \\
H(s,n) &:= \sum_{k=1}^{n} \frac{1}{k^s} \\
\zeta(s) &:= \sum_{k=1}^{\infty} \frac{1}{k^s}
\end{aligned}$$

3. Properties

Here are some properties that we will use throughout this paper:

Proposition 3.1. *For all real x with $|x| > 1$, we have:*

$$\log \frac{x}{x-1} = \sum_{k=1}^{\infty} \frac{1}{kx^k}$$

Proof. Because $|x| > 1$, we have $\frac{1}{|x|} < 1$. Therefore, we have the following:

$$\begin{aligned}
\frac{1}{1-\frac{1}{x}} &= \sum_{k=0}^{\infty} \frac{1}{x^k} \\
\frac{1}{x^2\left(1-\frac{1}{x}\right)} &= \sum_{k=2}^{\infty} \frac{1}{x^k} \\
\frac{1}{x(x-1)} &= \sum_{k=2}^{\infty} \frac{1}{x^k} \\
-\frac{1}{x(x-1)} &= -\sum_{k=2}^{\infty} \frac{1}{x^k} \\
\frac{1}{x} - \frac{1}{x-1} &= -\sum_{k=2}^{\infty} \frac{1}{x^k}
\end{aligned}$$

Now integrating both sides gives us:

$$\begin{aligned}
\int \left(\frac{1}{x} - \frac{1}{x-1}\right) dx &= -\int \sum_{k=2}^{\infty} \frac{1}{x^k} dx = -\sum_{k=2}^{\infty} \int \frac{1}{x^k} dx \\
\log x - \log(x-1) &= \sum_{k=1}^{\infty} \frac{1}{kx^k} \\
\log \frac{x}{x-1} &= \sum_{k=1}^{\infty} \frac{1}{kx^k}
\end{aligned}$$

This proves our proposition. □

Proposition 3.2. *For all real x the function $E^+(x) = \left(1+\frac{1}{x}\right)^x$ is increasing on the interval $[1, \infty)$.*

Proof. We have:

$$\begin{aligned}
E^+(x) &= \left(1+\frac{1}{x}\right)^x \\
\log E^+(x) &= x \log\left(1+\frac{1}{x}\right) \\
\frac{(E^+(x))'}{E^+(x)} &= \log\left(1+\frac{1}{x}\right) + x \cdot \frac{-\frac{1}{x^2}}{1+\frac{1}{x}} \\
&= \log\frac{x+1}{x} - \frac{1}{x+1}
\end{aligned}$$

By proposition 3.1, we have:

$$\begin{aligned}\frac{(E^+(x))'}{E^+(x)} &= \sum_{k=1}^{\infty} \frac{1}{k(x+1)^k} - \frac{1}{x+1} \\ &= \sum_{k=2}^{\infty} \frac{1}{k(x+1)^k} > 0\end{aligned}$$

Our proof is done because we have proved that $(E^+(x))' > 0$ for all $x \geq 1$. □

Using this proposition, we can establish the boundaries for $E^+(x)$ on $[1, \infty)$:

$$2 \leq E^+(x) < \lim_{x\to\infty} E^+(x) = e$$

Proposition 3.3. *For all real x the function $E^-(x) = \left(1 - \frac{1}{x}\right)^{-x}$ is decreasing on the interval $(1, \infty)$.*

Proof. We have:

$$\begin{aligned}E^-(x) &= \left(1 - \frac{1}{x}\right)^{-x} \\ \log E^-(x) &= -x \log\left(1 - \frac{1}{x}\right) \\ \frac{(E^-(x))'}{E^-(x)} &= -\log\left(1 - \frac{1}{x}\right) - x \cdot \frac{\frac{1}{x^2}}{1 - \frac{1}{x}} \\ &= -\log\frac{x-1}{x} - \frac{1}{x-1} \\ &= \log\frac{x}{x-1} - \frac{1}{x-1} \\ &= \sum_{k=1}^{\infty} \frac{1}{kx^k} - \frac{1}{x-1} \\ &< \sum_{k=1}^{\infty} \frac{1}{x^k} - \frac{1}{x-1} \\ &= \frac{1}{x-1} - \frac{1}{x-1} = 0\end{aligned}$$

Since $(E^-(x))' < 0$ for all $x > 1$, our proof is done. □

Note that this proposition can be used to establish the following inequality:

$$E^-(x) > \lim_{x\to\infty} E^-(x) = e$$

Proposition 3.4. *For every positive integer n, the function $g(n) := \log n - \sum_{k=1}^{n} \frac{1}{k}$ is increasing.*

Proof. We have:

$$\begin{aligned} g(n+1) - g(n) &= \log\left(\frac{n+1}{n}\right) - \frac{1}{n+1} \\ &= \sum_{k=1}^{\infty} \frac{1}{k(n+1)^k} - \frac{1}{n+1} \\ &= \sum_{k=2}^{\infty} \frac{1}{k(n+1)^k} > 0 \end{aligned}$$

This proves our proposition. □

4. The behavior of the gamma function near its points of discontinuity

Now let us analyze how the gamma function behaves near its points of discontinuity. First let's prove the following properties:

$$\lim_{x\to 0}\left(\frac{1}{x} - \Gamma(x)\right) = \gamma \tag{4.1}$$

$$\lim_{x\to 0}\left(-\frac{1}{x} - \Gamma(-x)\right) = \gamma \tag{4.2}$$

Proof. We have:

$$\begin{aligned} \lim_{x\to 0}\left(\frac{1}{x} - \Gamma(x)\right) &= -\lim_{x\to 0}\left(\Gamma(x) - \frac{1}{x}\right) = -\lim_{x\to 0}\left(\frac{\Gamma(1+x)}{x} - \frac{1}{x}\right) \\ &= -\lim_{x\to 0}\frac{\Gamma(1+x)-1}{x} = -\lim_{x\to 0}\frac{\Gamma(1+x)-\Gamma(1)}{x} \\ &= -\Gamma'(1) = \gamma \end{aligned}$$

Similarly:

$$\begin{aligned}
\lim_{x\to 0}\left(-\frac{1}{x}-\Gamma(-x)\right) &= -\lim_{x\to 0}\left(\frac{1}{x}+\Gamma(-x)\right)\\
&= -\lim_{x\to 0}\left(\frac{1}{x}+\frac{\Gamma(1-x)}{-x}\right)\\
&= -\lim_{x\to 0}\frac{1-\Gamma(1-x)}{x}\\
&= -\lim_{x\to 0}\frac{\Gamma(1)-\Gamma(1-x)}{x}\\
&= -\Gamma'(1)=\gamma
\end{aligned}$$

□

These limits tell us that the gamma function behaves almost exactly like $y=\frac{1}{x}$ when x gets close to 0. Combining these limits gives us:

$$\lim_{x\to 0}\left(\Gamma(x)+\Gamma(-x)\right) = -\lim_{x\to 0}\left[\left(\frac{1}{x}-\Gamma(x)\right)+\left(-\frac{1}{x}-\Gamma(-x)\right)\right] = -2\gamma$$

Now let's see how the gamma function behaves at other points of discontinuity. Using the recursive formula for gamma function gives us:

$$\begin{aligned}
\Gamma(x) &= \frac{\Gamma(x+n)}{\prod_{k=0}^{n-1}(x+k)}\\
(x+n)\Gamma(x) &= \frac{(x+n)\Gamma(x+n)}{\prod_{k=0}^{n-1}(x+k)}
\end{aligned}$$

Note that from (4.1) we have:

$$\lim_{x\to 0} x\Gamma(x) = \lim_{x\to 0}\left(x\left[\Gamma(x)-\frac{1}{x}\right]+1\right) = 0\cdot(-\gamma)+1=1$$

In this limit, it does not matter if x goes to zero from the left or right side. Therefore, we have: $\lim_{x\to -n^+}(x+n)\Gamma(x+n)=(-1)^n$, and so:

$$\lim_{x\to -n^+}(x+n)\Gamma(x) = \frac{(-1)^n}{n!}$$

Similarly:

$$\lim_{x\to -n^-}(x+n)\Gamma(x) = \frac{(-1)^n}{n!}$$

These two limit properties tell us that the gamma function behaves almost exactly as the function: $y = \frac{c}{x+n}$, where c is a constant depending on n, as x gets close to the point $-n$ either from the left or right hand side.

For any positive integer n, the limit properties of the gamma function show us that the graph is almost the same around $-n$. The difference in sign leads to the question of whether or not they cancel. We already have the answer for the point $x = 0$ with the limit we established earlier. They cancel each other, and so we are left with the constant -2γ. Our task now is to find the following limit (LM) (or to prove that it does not exist):

$$\lim_{x\to 0}(\Gamma(-n+x) + \Gamma(-n-x))$$

We have:

$$\begin{aligned}
LM &= \lim_{x\to 0}\left(\frac{\Gamma(x)}{\prod_{k=1}^{n}(x-k)} + \frac{\Gamma(-x)}{\prod_{k=1}^{n}(-x-k)}\right) \\
&= \lim_{x\to 0}\left(\frac{\Gamma(x)}{\prod_{k=1}^{n}(x-k)} + \frac{(-1)^n\Gamma(-x)}{\prod_{k=1}^{n}(x+k)}\right) \\
&= \lim_{x\to 0}\left(\frac{\Gamma(x)\prod_{k=1}^{n}(x+k) + (-1)^n\Gamma(-x)\prod_{k=1}^{n}(x-k)}{\prod_{k=1}^{n}(x^2-k^2)}\right) \\
&= \frac{(-1)^n}{(n!)^2}\cdot\lim_{x\to 0}\left(\Gamma(x)\prod_{k=1}^{n}(x+k) + (-1)^n\Gamma(-x)\prod_{k=1}^{n}(x-k)\right) \\
&= \frac{(-1)^n}{(n!)^2}\cdot\lim_{x\to 0}\sum_{k=0}^{n} c_k x^k \left(\Gamma(x) + (-1)^k\Gamma(-x)\right) \\
&= \frac{(-1)^n}{(n!)^2}\cdot\lim_{x\to 0}\sum_{k=0}^{n} c_k T_k
\end{aligned}$$

where c_k are expressions involving n, and $T_k = x^k\left(\Gamma(x) + (-1)^k\Gamma(-x)\right)$. Note that when k is even and greater than 0, we have:

$$\lim_{x\to 0} c_k x^k\left(\Gamma(x) + (-1)^k\Gamma(-x)\right) = c_k \cdot 0 \cdot (-2\gamma) = 0$$

So we are left with all the terms with k odd, and the term with $k = 0$, which we will take into account later. Let's look at the terms we have left (k odd):

$$\begin{aligned}
T_k &= x^k\left(\Gamma(x)-\Gamma(-x)\right) \\
&= x^k\left[\left(\Gamma(x)-\frac{1}{x}\right)+\left(-\frac{1}{x}-\Gamma(-x)\right)+\frac{2}{x}\right] \\
&= x^k\left(\frac{x\Gamma(x)-1}{x}+\frac{-1-x\Gamma(-x)}{x}\right)+2x^{k-1} \\
\lim_{x\to 0} T_k &= 0\cdot(-\gamma+\gamma)+2\lim_{x\to 0} x^{k-1} = 2\lim_{x\to 0} x^{k-1}
\end{aligned}$$

So T_k will go to zero if $k \geq 2$. That leaves us with only two terms left to consider. For the term T_1, we have:

$$c_1 = n!\left(1+\frac{1}{2}+\ldots+\frac{1}{n}\right)$$

And so:

$$\lim_{x\to 0} c_1 T_1 = 2n!\left(1+\frac{1}{2}+\ldots+\frac{1}{n}\right)$$

For the term with $k=0$ we have:

$$c_0 T_0 = n!\left(\Gamma(x)+\Gamma(-x)\right)$$

And so:

$$\lim_{x\to 0} c_0 T_0 = -2\gamma n!$$

Now we have enough information to calculate our original limit:

$$\begin{aligned}
LM &= \frac{(-1)^n}{(n!)^2}\cdot\lim_{x\to 0}\sum_{k=0}^{n} c_k T_k \\
&= \frac{(-1)^n}{(n!)^2}\cdot\lim_{x\to 0}\left(c_0T_0+c_1T_1\right) \\
&= \frac{(-1)^n}{(n!)^2}\cdot\left(-2\gamma n!+2n!\left(1+\frac{1}{2}+\ldots+\frac{1}{n}\right)\right)
\end{aligned}$$

Finally:

$$\lim_{x\to 0}\left(\Gamma(-n+x)+\Gamma(-n-x)\right)=\frac{(-1)^n(2H(1,n)-2\gamma)}{n!}$$

5. Critical points of the Gamma function on the left

We will analyze the critical points of the gamma function on each one-unit-wide interval: $(-1,0)$, $(-2,-1)$, $(-3,-2)$,.. and so forth. For the n^{th} interval $(-n,-n+1)$, let: $x_n = x_0 + n$, where x_0 is the x-coordinate of the critical point of the function on this interval ($\Gamma'(x_0) =$

0). Note that because $x_0 \in (-n, -n+1)$, we have $x_n \in (0,1)$. We will explain later how we only have one critical point per interval. Now:

$$\begin{aligned}
\Gamma(x) &= (x-1)\Gamma(x-1) \\
\Gamma'(x) &= \Gamma(x-1) + (x-1)\Gamma'(x-1) \\
\Gamma'(x) &= \frac{\Gamma(x)}{x-1} + \frac{\Gamma(x)}{\Gamma(x-1)} \cdot \Gamma'(x-1) \\
\frac{\Gamma'(x)}{\Gamma(x)} &= \frac{1}{x-1} + \frac{\Gamma'(x-1)}{\Gamma(x-1)} \\
\Psi(x) &= \frac{1}{x-1} + \Psi(x-1)
\end{aligned}$$

Hence:

$$\Psi(x) = \sum_{k=1}^{n} \frac{1}{x-k} + \Psi(x-n)$$

Let x_0 be the x-coordinate of the critical point in the n^{th} interval, and in the above equation let $x - n = x_0$. That means $x = x_0 + n = x_n$, as defined earlier. Therefore, we have:

$$\Psi(x_n) = \sum_{k=1}^{n} \frac{1}{x_n - k}$$

because $\Psi(x-n) = \Psi(x_0) = \frac{\Gamma'(x_0)}{\Gamma(x_0)} = 0$. In other words, x_n is a root of the equation:

$$\Psi(x) = \sum_{k=1}^{n} \frac{1}{x-k} \tag{5.1}$$

We will prove that (5.1) has only one root on $(0,1)$, which means the value x_n is unique, as is x_0. Let's look at our recursive formula for $\Psi(x)$:

$$\begin{aligned}
\Psi(x) &= \frac{1}{x-1} + \Psi(x-1) \\
\Psi(x+1) &= \frac{1}{x} + \Psi(x) \\
\Psi(x+1) - \frac{1}{x} &= \Psi(x) \\
\lim_{x \to 0^+} \left[\Psi(x+1) - \frac{1}{x} \right] &= \lim_{x \to 0^+} \Psi(x) \\
\Psi(1) - \infty &= \lim_{x \to 0^+} \Psi(x)
\end{aligned}$$

with $\Psi(1) = -\gamma$, a constant. Therefore:

$$\lim_{x \to 0^+} \Psi(x) = -\infty$$

Now let's take the first derivative of $\Psi(x)$ using the following formula for the digamma function:

$$\begin{aligned}
\Psi(x) &= -\gamma + \sum_{n=1}^{\infty} \frac{x-1}{n(n+x-1)} \\
\Psi'(x) &= \sum_{n=1}^{\infty} \frac{1}{(n+x-1)^2} > 0
\end{aligned}$$

This means that $\Psi(x)$ is an increasing function, so when x goes from 0 to 1, $\Psi(x)$ goes from $-\infty$ to $\Psi(1) = -\gamma$. On the other hand:

$$\frac{d}{dx}\left[\frac{1}{x-k}\right] = -\frac{1}{(x-k)^2} < 0,$$

which means that the right side of (5.1) is decreasing with respect to x. When $x = 0$, our right side (RS) becomes:

$$RS = \sum_{k=1}^{n} \frac{-1}{k} = -H(1,n)$$

Moreover:

$$\begin{aligned}
\lim_{x \to 1^-} RS &= \lim_{x \to 1^-} \frac{1}{x-1} + \lim_{x \to 1^-} \sum_{k=2}^{n} \frac{1}{x-k} \\
&= -\infty + \sum_{k=1}^{n-1} \frac{-1}{k} \\
&= -\infty - constant \\
\lim_{x \to 1^-} RS &= -\infty
\end{aligned}$$

Now let's summarize what we have found. As x goes from 0 to 1, our left side increases from $-\infty$ to $-\gamma$, and our right side decreases from $-H(1,n)$ to $-\infty$. This fact guarantees us exactly one solution to our equation.

Assume that we have found the root x_n to our equation. We will now try to see whether x_{n+1} is smaller or greater than x_n. Note that in our equation, if we start with $x = 0$ and keep increasing the value of x searching for our root, we will have our left side is smaller than our right side. That means when we come to a value of x where we have our left side greater than our right side, we know that we have "passed" our root, meaning our root is smaller than the value of x we are at right now. Our x_{n+1} is the root of the following equation:

$$\Psi(x) = \sum_{k=1}^{n+1} \frac{1}{x-k}$$

Note that when $x = x_n$, we have:

$$RS = \sum_{k=1}^{n} \frac{1}{x_n - k} + \frac{1}{x_n - n - 1} < \sum_{k=1}^{n} \frac{1}{x_n - k} = \Psi(x_n) = LS$$

So at the value $x = x_n$, we have our left side greater than our right side. This tells us that our root x_{n+1} is smaller than x_n. Now we have a strictly decreasing sequence $\{x_n\}$.

Now let's go back to our equation (5.1) and multiply both sides by x. We have:

$$\begin{aligned} x\Psi(x) &= \sum_{k=1}^{n} \frac{x}{x-k} = \sum_{k=1}^{n} \left(1 + \frac{k}{x-k}\right) \\ &= n + \sum_{k=1}^{n} \frac{1}{\frac{x}{k} - 1} = n - \sum_{k=1}^{n} \frac{1}{1 - \frac{x}{k}} \\ &= n - \sum_{k=1}^{n} \sum_{l=0}^{\infty} \left(\frac{x}{k}\right)^l \\ &= - \sum_{k=1}^{n} \sum_{l=1}^{\infty} \left(\frac{x}{k}\right)^l \\ &= - \sum_{l=1}^{\infty} \left[x^l \cdot \sum_{k=1}^{n} \frac{1}{k^l}\right] \end{aligned}$$

Now changing the notation gives us:

$$\begin{aligned}
x\Psi(x) &= -\sum_{l=1}^{\infty}\left[x^l \cdot H(l,n)\right] \\
&= -x \cdot H(1,n) - \sum_{l=2}^{\infty}\left[x^l \cdot H(l,n)\right] \\
&= -x\left(\log n + H(1,n) - \log n\right) - \sum_{l=2}^{\infty}\left[x^l \cdot H(l,n)\right] \\
&= -x\log n - x\left(H(1,n) - \log n\right) - \sum_{l=2}^{\infty}\left[x^l \cdot H(l,n)\right] \\
x\Psi(x) &= -x\log n - A
\end{aligned}$$

with:

$$A = x\left(H(1,n) - \log n\right) + \sum_{l=2}^{\infty}\left[x^l \cdot H(l,n)\right]$$

According to proposition 3.4, we have $H(1,n) - \log n = \sum_{k=1}^{n} \frac{1}{k} - \log n$ is a decreasing function, so:

$$H(1,n) - \log n > \lim_{n\to\infty}\left(H(1,n) - \log n\right) = \gamma > 0$$

and also:

$$H(1,n) - \log n < H(1,1) - \log 1 = 1.$$

Also notice that:

$$H(s,n) = \sum_{k=1}^{n} \frac{1}{k^s} < \sum_{k=1}^{\infty} \frac{1}{k^s} = \zeta(s)$$

Therefore:

$$\begin{aligned}
A &= x\left(H(1,n) - \log n\right) + \sum_{l=2}^{\infty}\left[x^l \cdot H(l,n)\right] \\
A &< x + \sum_{l=2}^{\infty} x^l \zeta(l) \\
&< x\zeta(2) + \sum_{l=2}^{\infty} x^l \zeta(2) \\
&= \zeta(2)\sum_{l=1}^{\infty} x^l \\
&= \zeta(2) \cdot \frac{x}{1-x}
\end{aligned}$$

The above inequality is true because $1 < \zeta(2) = \frac{\pi^2}{6}$, and $\zeta(2) > \zeta(l)$ for all $l \geq 3$. Remember that we are using the equation corresponding to x_n, and $x_n < x_1$. Therefore, we have:

$$A < \zeta(2) \cdot \frac{x_n}{1 - x_n} < \zeta(2) \cdot \frac{x_1}{1 - x_1}$$

because the function $f(x) = \frac{x}{1-x}$ is increasing on $(0, 1)$. We have proved that there is exactly one critical point in each interval. That means x_1 can be considered a constant, which means A is bounded. Next we will show that $x\Psi(s)$ is also bounded on $(0, 1)$. From our recursive formula for $\Psi(x)$, we have:

$$\begin{aligned}
\Psi(x+1) &= \Psi(x) + \frac{1}{x} \\
x\Psi(x+1) &= x\Psi(x) + 1 \\
\lim_{x\to 0} x\Psi(x+1) &= \lim_{x\to 0} x\Psi(x) + 1 \\
0 &= \lim_{x\to 0} x\Psi(x) + 1 \\
\lim_{x\to 0} x\Psi(x) &= -1
\end{aligned}$$

The above argument is valid because $\Psi(1) = -\gamma$, a constant. Now the function $x\Psi(x)$ is continuous on $(0, 1)$, and $\lim_{x\to 0} x\Psi(x) = -1$ and its value at $x = 1$ is $-\gamma$. Therefore, $x\Psi(x)$ is bounded on $(0, 1)$. Using these results gives us:

$$-x_n \log n = x_n\Psi(x_n) + A$$

is also bounded. Now if n goes to infinity, then in order for $x_n \log n$ to be bounded, we must have:

$$\lim_{n\to\infty} x_n = 0$$

We showed earlier that $H(1, n) - \log n > 0$. That means $A > 0$. Now we have two boundaries for A:

$$0 < A < \zeta(2) \cdot \frac{x_n}{1 - x_n}$$

However, $\lim_{n\to\infty} \frac{x_n}{1-x_n} = \lim_{x\to 0} \frac{x}{1-x} = 0$. This means that:

$$\lim_{n\to\infty} A = 0$$

Therefore:

$$\begin{aligned}
-\lim_{n\to\infty} x_n \log n &= \lim_{n\to\infty} x_n\Psi(x_n) + \lim_{n\to\infty} A \\
-\lim_{n\to\infty} x_n \log n &= -1 + 0 \\
\lim_{n\to\infty} x_n \log n &= 1
\end{aligned}$$

Finally, we have established a relationship between x_n and n, through a limit property.

Now we will try to do the same thing with the value of the gamma function at those critical points. We have:

$$\begin{aligned}\Gamma(x_n) &= (x_n-1)\Gamma(x_n-1)\\ &= (x_n-1)(x_n-2)\Gamma(x_n-2)\\ &\vdots\\ \Gamma(x_n) &= \Gamma(x_n-n)\prod_{k=1}^{n}(x_n-k)\end{aligned}$$

Let d_n be the value of the gamma function at the critical point in the n^{th} interval. Note that with the definition of x_n, then $x_n - n$ is the x-coordinate for the critical point on the n^{th} interval. Therefore:

$$d_n = \Gamma(x_n - n) = \frac{\Gamma(x_n)}{\prod_{k=1}^{n}(x_n-k)}$$

Let's look at the following expression:

$$B = \frac{|\prod_{k=1}^{n}(x_n-k)|}{n!}$$

We have:

$$\begin{aligned}B &= \frac{\prod_{k=1}^{n}(k-x_n)}{n!} = \prod_{k=1}^{n}\left(\frac{k-x_n}{k}\right)\\ &= \prod_{k=1}^{n}\left(1-\frac{x_n}{k}\right) = \prod_{k=1}^{n}\left[\left(1-\frac{1}{\frac{k}{x_n}}\right)^{-\frac{k}{x_n}}\right]^{-\frac{x_n}{k}}\\ &= \prod_{k=1}^{n}\left(E^{-}\left(\frac{k}{x_n}\right)\right)^{-\frac{x_n}{k}} = \prod_{k=1}^{n}(e+\epsilon_k)^{-\frac{x_n}{k}}\\ &= \left(\prod_{k=1}^{n}(e+\epsilon_k)^{\frac{x_n}{k}}\right)^{-1} = C^{-1}\end{aligned}$$

with:

$$\epsilon_k = E^{-}\left(\frac{k}{x_n}\right) - e > 0$$

Now:

$$\begin{aligned}
C &= \prod_{k=1}^{n}(e+\epsilon_k)^{\frac{x_n}{k}} = \prod_{k=1}^{n}\left[\left(\frac{e+\epsilon_k}{e}\right)^{\frac{x_n}{k}}\cdot e^{\frac{x_n}{k}}\right] \\
&= \prod_{k=1}^{n}\left[\left(1+\frac{\epsilon_k}{e}\right)^{\frac{x_n}{k}}\cdot e^{\frac{x_n}{k}}\right] = \prod_{k=1}^{n}\left[\left(\left(1+\frac{1}{\frac{e}{\epsilon_k}}\right)^{\frac{e}{\epsilon_k}}\right)^{\frac{\epsilon_k x_n}{ek}}\cdot e^{\frac{x_n}{k}}\right] \\
&= \prod_{k=1}^{n}\left[\left(E^{+}\left(\frac{e}{\epsilon_k}\right)\right)^{\frac{\epsilon_k x_n}{ek}}\cdot e^{\frac{x_n}{k}}\right] \\
\log C &= \sum_{k=1}^{n}\frac{\epsilon_k x_n}{ek}\log E^{+}\left(\frac{e}{\epsilon_k}\right)+\sum_{k=1}^{n}\frac{x_n}{k}
\end{aligned}$$

Since $\frac{1}{x_n} < \frac{2}{x_n} < ... < \frac{n}{x_n}$, we have $E^{-}\left(\frac{1}{x_n}\right) > E^{-}\left(\frac{2}{x_n}\right) > ... > E^{-}\left(\frac{n}{x_n}\right)$, and $\epsilon_1 > \epsilon_2 > ... > \epsilon_n$. Using the fact that $E^{+}(x) > 2$ for all $x > 1$, we have:

$$\begin{aligned}
\log C &> \sum_{k=1}^{n}\frac{\epsilon_k x_n}{ek}\log 2+\sum_{k=1}^{n}\frac{x_n}{k} \\
&= \sum_{k=1}^{n}\frac{\epsilon_n x_n\log 2}{ek}+\sum_{k=1}^{n}\frac{x_n}{k} \\
&= \frac{\epsilon_n\log 2}{e}\cdot x_n\sum_{k=1}^{n}\frac{1}{k}+x_n\sum_{k=1}^{n}\frac{1}{k} \\
&= \frac{\epsilon_n\log 2}{e}\cdot x_n H(1,n)+x_n H(1,n) \\
&= \frac{\epsilon_n\log 2}{e}\cdot x_n\left(H(1,n)-\log n+\log n\right)+ \\
&\quad +x_n\left(H(1,n)-\log n+\log n\right) \\
&= \frac{\epsilon_n\log 2}{e}\cdot x_n\left(H(1,n)-\log n\right)+\frac{\epsilon_n\log 2}{e}\cdot x_n\log n \\
&\quad +x_n\left(H(1,n)-\log n\right)+x_n\log n \\
&= D
\end{aligned}$$

Note that:

$$\lim_{n\to\infty}\epsilon_n = 0 = \lim_{n\to\infty}x_n$$
$$\lim_{n\to\infty}x_n\log n = 1$$
$$\lim_{n\to\infty}\left(H(1,n)-\log n\right) = \gamma$$

Using these limits gives us:

$$\lim_{n\to\infty} D = 0 \cdot \gamma + 0 \cdot 1 + 0 \cdot \gamma + 1 = 1$$

Since $E^+(x) < e$ for all $x > 1$, we have:

$$\begin{aligned}
\log C &< \sum_{k=1}^{n} \frac{\epsilon_k x_n}{ek} + \sum_{k=1}^{n} \frac{x_n}{k} \\
&= \sum_{k=1}^{n} \frac{\epsilon_1 x_n}{ek} + \sum_{k=1}^{n} \frac{x_n}{k} \\
&= \frac{\epsilon_1}{e} \cdot x_n \sum_{k=1}^{n} \frac{1}{k} + x_n \sum_{k=1}^{n} \frac{1}{k} \\
&= \frac{\epsilon_1}{e} \cdot x_n H(1,n) + x_n H(1,n) \\
&= \frac{\epsilon_1}{e} \cdot x_n \left(H(1,n) - \log n + \log n\right) + \\
&\quad + x_n \left(H(1,n) - \log n + \log n\right) \\
&= \frac{\epsilon_1}{e} \cdot x_n \left(H(1,n) - \log n\right) + \frac{\epsilon_1}{e} \cdot x_n \log n \\
&\quad + x_n \left(H(1,n) - \log n\right) + x_n \log n \\
&= F
\end{aligned}$$

Note that when n goes to infinity, ϵ_1 also goes to zero. Therefore, we have:

$$\lim_{n\to\infty} F = 0 \cdot 0 \cdot \gamma + 0 \cdot 1 + 0 \cdot \gamma + 1 = 1$$

Now we have $D < \log C < F$ and $\lim_{n\to\infty} D = \lim_{n\to\infty} F = 1$. Therefore:

$$\lim_{n\to\infty} \log C = 1$$

And so:

$$\lim_{n\to\infty} C = e$$

That means:

$$\lim_{n\to\infty} \frac{\left|\prod_{k=1}^{n} (x_n - k)\right|}{n!} = \lim_{n\to\infty} B = \frac{1}{e}$$

Applying this result gives us:

$$\begin{aligned}
|d_n| &= \frac{\Gamma(x_n)}{|\prod_{k=1}^{n}(x_n-k)|} = \frac{\Gamma(x_n+1)}{x_n\cdot|\prod_{k=1}^{n}(x_n-k)|} \\
\frac{n!|d_n|}{\log n} &= \Gamma(x_n+1)\cdot\frac{1}{x_n\log n}\cdot\frac{n!}{|\prod_{k=1}(x_n-k)|} \\
\lim_{n\to\infty}\frac{n!|d_n|}{\log n} &= 1\cdot\frac{1}{1}\cdot e \\
\lim_{n\to\infty}\frac{n!|d_n|}{\log n} &= e
\end{aligned}$$

And now we have established a relationship between the value of the gamma function at its critical point in the n^{th} interval and the value of n.

6. The bluntness of the gamma function on the negative side

In this section, we will try to explain why the graph of the gamma function seems to flatten out when we move to the left from one interval to the next. In order to do that, we consider solving the following equation in each interval:

$$|\Gamma'(x)| = \alpha \tag{6.1}$$

for some real and positive α. Let $x_k^* - k$ denote the solution to (6.1) on the corresponding interval $(-k, -k+1)$. We are just interested in finding the root that is on the right side of the critical point of that interval. In other words, if we are solving the equation (6.1) for the n^{th} interval, then we only look for the root x_n^* that qualifies: $x_n^* > x_n$, where x_n is the critical point of the gamma function in the n^{th} interval. We have shown earlier that:

$$\Gamma'(x) = \Gamma(x-1) + (x-1)\Gamma'(x-1)$$

Using this properties n times gives us:

$$\Gamma'(x) = \Gamma(x)\sum_{k=1}^{n}\frac{1}{x-k} + \Gamma'(x-n)\prod_{k=1}^{n}(x-k).$$

Therefore:

$$\Gamma'(x-n) = \frac{\Gamma'(x) + \Gamma(x)\sum_{k=1}^{n}\frac{1}{k-x}}{\prod_{k=1}^{n}(x-k)}$$

Now assume that our sequence $\{x_n^*\}$ is bounded by a positive constant s less than 1. In the other words: $x_n^* \le s < 1$, for all positive integers n. We will prove that when n is large enough, we cannot find a solution for equation (6.1) in the n^{th} interval. Of course this contradicts with the fact that the domain of $\Gamma'(x)$ is from negative infinity to positive infinity on any intervals on the negative side. This contradiction would confirm that when n is large enough, the graph of the gamma function will look as flattened as possible.

In order to prove that we will not have an solution (on the right side of the critical point) to (6.1) when n is large enough, it suffices to show that: $|\Gamma'(s-n)| < \alpha$, when n is sufficiently large. In the equation we derived earlier, substituting x by s gives us:

$$|\Gamma'(s-n)| = \left| \frac{\Gamma'(s) + \Gamma(s)\sum_{k=1}^{n} \frac{1}{k-s}}{\prod_{k=1}^{n}(s-k)} \right|$$

We will analyze the numerator (N) and the denominator (D) separately. Note that since s is a positive constant less than 1, we have $\Gamma'(s)$, $\Gamma(s)$, and $(1-s)$ are constants also. We have:

$$\begin{aligned} N &= \Gamma'(s) + \Gamma(s)\sum_{k=1}^{n} \frac{1}{k-s} \\ &= \Gamma'(s) + \frac{\Gamma(s)}{1-s} + \Gamma(s)\sum_{k=2}^{n} \frac{1}{k-1+(1-s)} \\ &< \Gamma'(s) + \frac{\Gamma(s)}{1-s} + \Gamma(s)\sum_{k=2}^{n} \frac{1}{k-1} \\ &= \Gamma'(s) + \frac{\Gamma(s)}{1-s} + \Gamma(s)H(1, n-1) \end{aligned}$$

Therefore:

$$\lim_{n\to\infty} \frac{|N|}{\log n} \leq \lim_{n\to\infty} \frac{H(1, n-1)}{\log n} = 1$$

Now for our denominator, we have:

$$\begin{aligned} |D| &= \prod_{k=1}^{n}(k-s) = (1-s)\prod_{k=2}^{n}(k-1+(1-s)) \\ &> (1-s)\prod_{k=2}^{n}(k-1) = (1-s)(n-1)! \\ \frac{|D|}{(n-1)!} &> 1-s \end{aligned}$$

Combining these results gives us:

$$\lim_{n\to\infty} \frac{\frac{|N|}{\log n}}{\frac{|D|}{(n-1)!}} < \frac{1}{1-s}$$

or:

$$\lim_{n\to\infty} \left(\frac{|N|}{|D|} \cdot \frac{(n-1)!}{\log n} \right) < \frac{1}{1-s}$$

However: $\lim_{n\to\infty} \frac{(n-1)!}{\log n} = \infty$. This tells us that:

$$\lim_{n\to\infty} \frac{|N|}{|D|} = 0$$

In other words: $\lim_{n\to\infty} |\Gamma'(s-n)| = 0$. Note that what we have proved is pretty powerful. It tells us that no matter how close s is to 1, we always have $\Gamma'(s-n)$ as close to zero as possible when n is large enough. Equivalently:

$$\lim_{n\to\infty} x_n^* = 1.$$

Our next goal is to analyze how fast x_n^* goes to 1. We will do that by analyzing the sequence $\{y_n\}$ defined as: $y_n = 1 - x_n^*$. Since $\lim_{n\to\infty} x_n^* = 1$, we have: $\lim_{n\to\infty} y_n = 0$. We have:

$$|\Gamma'(x_n^* - n)| = \left| \frac{\Gamma'(x_n^*) + \Gamma(x_n^*) \sum_{k=1}^{n} \left(\frac{1}{k - x_n^*} \right)}{\prod_{k=1}^{n} (k - x_n^*)} \right| = \alpha.$$

Changing the variable gives us:

$$\begin{aligned} |\Gamma'(x_n^* - n)| = \left| \frac{\Gamma'(1-y_n) + \Gamma(1-y_n) \sum_{k=0}^{n-1} \left(\frac{1}{y_n + k} \right)}{\prod_{k=0}^{n-1} (y_n + k)} \right| &= \alpha \\ \left| \frac{\Gamma'(1-y_n) + \frac{\Gamma(1-y_n)}{y_n} + \Gamma(1-y_n) \sum_{k=1}^{n-1} \left(\frac{1}{y_n + k} \right)}{\prod_{k=0}^{n-1} (y_n + k)} \right| &= \alpha \\ \left| \frac{\Gamma'(1-y_n)}{\prod_{k=0}^{n-1} (y_n + k)} + \frac{\Gamma(1-y_n)}{y_n \prod_{k=0}^{n-1} (y_n + k)} + \frac{\Gamma(1-y_n) \sum_{k=1}^{n-1} \left(\frac{1}{y_n + k} \right)}{\prod_{k=0}^{n-1} (y_n + k)} \right| &= \alpha \\ |A + B + C| &= \alpha, \end{aligned}$$

with A, B, and C are the first, second, and third term respectively. We will take the limit when n goes to infinity for both sides. Notice that:

$$\lim_{n\to\infty} \Gamma'(1-y_n) = \Gamma'(1) = -\gamma,$$

and:

$$\lim_{n\to\infty} \Gamma(1-y_n) = \Gamma(1) = 1.$$

The first thing we can easily notice is that: $\lim_{n\to\infty} A = 0$. Dealing with B and C requires us to prove something first. Recall that when we have the sequence $\{x_n\}$ that qualifies: $0 < x_n < 1$, and $\lim_{n\to\infty} x_n \log n = 1$, we have:

$$\lim_{n\to\infty} \frac{\prod_{k=1}^{n}(k-x_n)}{n!} = \frac{1}{e}$$

When we look back at the proof, we can see that if we had $\lim_{n\to\infty} x_n \log n = c$ instead of 1, we would have:

$$\lim_{n\to\infty} \frac{\prod_{k=1}^{n}(k-x_n)}{n!} = \frac{1}{e^c}$$

Now let's look at the following limit:

$$\begin{aligned}
\lim_{n\to\infty} \left(\frac{\prod_{k=1}^{n}(k-x_n)}{n!} \cdot \frac{\prod_{k=1}^{n}(k+x_n)}{n!} \right) &= \lim_{n\to\infty} \frac{\prod_{k=1}^{n}(k^2-x_n^2)}{(n!)^2} \\
&= \lim_{n\to\infty} \prod_{k=1}^{n} \left(\frac{k^2-x_n^2}{k^2} \right) \\
&= \lim_{n\to\infty} \prod_{k=1}^{n} \left(1 - \left(\frac{x_n}{k} \right)^2 \right) \\
&= \lim_{n\to\infty} \frac{sin(\pi x_n)}{\pi x_n} \\
&= 1
\end{aligned}$$

From the limit we just derived, we have:

$$\lim_{n\to\infty} \frac{\prod_{k=1}^{n}(k+x_n)}{n!} = e^c,$$

Note that the second to last equal sign should be justified more formally, or we can use the similar proof that we did for the other case. Assume we have $\lim_{n\to\infty} x_n \log n = c$. Let:

$$G = \frac{|\prod_{k=1}^{n}(x_n+k)|}{n!}$$

We have:

$$\begin{aligned}
B &= \frac{\prod_{k=1}^{n}(k+x_n)}{n!} = \prod_{k=1}^{n}\left(\frac{k+x_n}{k}\right) \\
&= \prod_{k=1}^{n}\left(1+\frac{x_n}{k}\right) = \prod_{k=1}^{n}\left[\left(1+\frac{1}{\frac{k}{x_n}}\right)^{\frac{k}{x_n}}\right]^{\frac{x_n}{k}} \\
&= \prod_{k=1}^{n}\left(E^{+}\left(\frac{k}{x_n}\right)\right)^{\frac{x_n}{k}} \\
&= \prod_{k=1}^{n}(e-\epsilon_k)^{\frac{x_n}{k}} = H
\end{aligned}$$

with:

$$\epsilon_k = e - E^{+}\left(\frac{k}{x_n}\right) > 0$$

Now:

$$\begin{aligned}
H &= \prod_{k=1}^{n}(e-\epsilon_k)^{\frac{x_n}{k}} = \prod_{k=1}^{n}\left[\left(\frac{e-\epsilon_k}{e}\right)^{\frac{x_n}{k}} \cdot e^{\frac{x_n}{k}}\right] \\
&= \prod_{k=1}^{n}\left[\left(1-\frac{\epsilon_k}{e}\right)^{\frac{x_n}{k}} \cdot e^{\frac{x_n}{k}}\right] = \prod_{k=1}^{n}\left[\left(\left(1-\frac{1}{\frac{e}{\epsilon_k}}\right)^{-\frac{e}{\epsilon_k}}\right)^{\frac{-\epsilon_k x_n}{ek}} \cdot e^{\frac{x_n}{k}}\right] \\
&= \prod_{k=1}^{n}\left[\left(E^{-}\left(\frac{e}{\epsilon_k}\right)\right)^{\frac{-\epsilon_k x_n}{ek}} \cdot e^{\frac{x_n}{k}}\right] \\
\log H &= \sum_{k=1}^{n}\frac{-\epsilon_k x_n}{ek}\log E^{-}\left(\frac{e}{\epsilon_k}\right) + \sum_{k=1}^{n}\frac{x_n}{k}
\end{aligned}$$

Since $\frac{1}{x_n} < \frac{2}{x_n} < ... < \frac{n}{x_n}$, we have $E^{+}\left(\frac{1}{x_n}\right) < E^{+}\left(\frac{2}{x_n}\right) < ... < E^{+}\left(\frac{n}{x_n}\right)$, and $\epsilon_1 > \epsilon_2 > ... > \epsilon_n$. We have $\frac{k}{x_n} \geq \frac{1}{x_n} \geq 1$. Therefore, $\epsilon_k = e - E^{+}\left(\frac{k}{x_n}\right) < e - E^{+}(1) = e - 2 < 1$. Hence, $\frac{e}{\epsilon_k} > e > 2$.

Therefore, $E^{-}\left(\frac{e}{\epsilon_k}\right) < E^{-}(2) = 4$. We have:

$$\begin{aligned}
\log H &> \sum_{k=1}^{n} -\frac{\epsilon_k x_n}{ek}\log 4 + \sum_{k=1}^{n}\frac{x_n}{k} \\
&> \sum_{k=1}^{n} -\frac{\epsilon_1 x_n \log 4}{ek} + \sum_{k=1}^{n}\frac{x_n}{k} \\
&= -\frac{\epsilon_1 \log 4}{e}\cdot x_n \sum_{k=1}^{n}\frac{1}{k} + x_n \sum_{k=1}^{n}\frac{1}{k} \\
&= -\frac{\epsilon_1 \log 4}{e}\cdot x_n H(1,n) + x_n H(1,n) \\
&= -\frac{\epsilon_1 \log 4}{e}\cdot x_n\left(H(1,n) - \log n + \log n\right) + \\
&\quad + x_n\left(H(1,n) - \log n + \log n\right) \\
&= -\frac{\epsilon_1 \log 4}{e}\cdot x_n\left(H(1,n) - \log n\right) - \frac{\epsilon_1 \log 4}{e}\cdot x_n \log n \\
&\quad + x_n\left(H(1,n) - \log n\right) + x_n \log n \\
&= D
\end{aligned}$$

Note that:

$$\lim_{n\to\infty} \epsilon_1 = 0 = \lim_{n\to\infty} x_n$$

$$\lim_{n\to\infty} x_n \log n = c$$

$$\lim_{n\to\infty} \left(H(1,n) - \log n\right) = \gamma$$

Using these limits gives us:

$$\lim_{n\to\infty} D = 0\cdot\gamma + 0\cdot c + 0\cdot\gamma + c = c$$

Since $E^{-}(x) > e$ for all $x > 1$, we have:

$$\begin{aligned}
\log H &< -\sum_{k=1}^{n} \frac{\epsilon_k x_n}{ek} + \sum_{k=1}^{n} \frac{x_n}{k} \\
&= -\sum_{k=1}^{n} \frac{\epsilon_n x_n}{ek} + \sum_{k=1}^{n} \frac{x_n}{k} \\
&= -\frac{\epsilon_n}{e} \cdot x_n \sum_{k=1}^{n} \frac{1}{k} + x_n \sum_{k=1}^{n} \frac{1}{k} \\
&= -\frac{\epsilon_n}{e} \cdot x_n H(1,n) + x_n H(1,n) \\
&= -\frac{\epsilon_n}{e} \cdot x_n \left(H(1,n) - \log n + \log n\right) + \\
&\quad + x_n \left(H(1,n) - \log n + \log n\right) \\
&= -\frac{\epsilon_n}{e} \cdot x_n \left(H(1,n) - \log n\right) - \frac{\epsilon_n}{e} \cdot x_n \log n \\
&\quad + x_n \left(H(1,n) - \log n\right) + x_n \log n \\
&= F
\end{aligned}$$

Note that when n goes to infinity, ϵ_n also goes to zero. Therefore, we have:

$$\lim_{n\to\infty} F = 0 \cdot 0 \cdot \gamma - 0 \cdot c + 0 \cdot \gamma + c = c$$

Now we have $D < \log H < F$ and $\lim_{n\to\infty} D = \lim_{n\to\infty} F = c$. Therefore:

$$\lim_{n\to\infty} \log H = c$$

And so:

$$\lim_{n\to\infty} H = e^c$$

That means:

$$\lim_{n\to\infty} \frac{|\prod_{k=1}^{n}(x_n + k)|}{n!} = \lim_{n\to\infty} G = e^c$$

Now let's go back to analyze our terms B and C. Notice that:

$$\prod_{k=1}^{n-1}(y_n + k) > \prod_{k=1}^{n-1} k = (n-1)!$$

and:

$$\sum_{k=2}^{n} \frac{1}{k} < \sum_{k=1}^{n-1} \left(\frac{1}{y_n + k}\right) < \sum_{k=1}^{n-1} \frac{1}{k}$$

Therefore:

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n-1}\left(\frac{1}{y_n+k}\right)}{\log n} = 1$$

We have:

$$\begin{aligned}
|A+B+C| &= \alpha \\
\left|\frac{A}{C}+\frac{B}{C}+1\right| &= \frac{\alpha}{C} \\
\lim_{n\to\infty}\left|\frac{1}{y_n\sum_{k=1}^{n-1}\left(\frac{1}{y_n+k}\right)}+1\right| &= \infty \\
\lim_{n\to\infty} y_n\sum_{k=1}^{n-1}\left(\frac{1}{y_n+k}\right) &= 0 \\
\lim_{n\to\infty}\left((y_n\log n)\cdot\frac{\sum_{k=1}^{n-1}\left(\frac{1}{y_n+k}\right)}{\log n}\right) &= 0 \\
\lim_{n\to\infty}(y_n\log n)\cdot 1 &= 0 \\
\lim_{n\to\infty} y_n\log n = 0
\end{aligned}$$

Using this limit gives us:

$$\lim_{n\to\infty}\frac{\prod_{k=1}^{n}(k-y_n)}{n!} = \frac{1}{e^0} = 1$$

and:

$$\lim_{n\to\infty}\frac{\prod_{k=1}^{n}(k+y_n)}{n!} = e^0 = 1$$

From our equation: $|A+B+C| = \alpha$, multiplying both sides by $y_n\prod_{k=0}^{n-1}$ and taking the limit when n goes to infinity gives us:

$$\begin{aligned}
|0+1+0| &= \lim_{n\to\infty}\alpha y_n^2\prod_{k=1}^{n-1}(y_n+k) \\
\lim_{n\to\infty}\left(\frac{y_n^2}{y_n+n}\prod_{k=1}^{n}(y_n+k)\right) &= \frac{1}{\alpha} \\
\lim_{n\to\infty}\left(y_n^2(n-1)!\cdot\frac{n}{y_n+n}\cdot\frac{\prod_{k=1}^{n}(y_n+k)}{n!}\right) &= \frac{1}{\alpha} \\
\lim_{n\to\infty} y_n^2(n-1)! &= \frac{1}{\alpha}
\end{aligned}$$

Now we will use this limit property to prove that for n sufficiently large, the sequence $\{x_n^*\}$ is strictly increasing. Using the recursive formula for the first derivative of the gamma function n times gives us:

$$\Gamma'(x_n^*) = \Gamma(x_n^*) \sum_{k=1}^{n} \frac{1}{x_n^* - k} + \alpha \prod_{k=1}^{n} |x_n^* - k|$$

After dividing both sides by $\Gamma(x_n^*)$, we have:

$$\Psi(x_n^*) = \sum_{k=1}^{n} \frac{1}{x_n^* - k} + \alpha \cdot \frac{\prod_{k=1}^{n} |x_n^* - k|}{\Gamma(x_n^*)}$$

Moreover, $\Gamma(x_n^*) = \Gamma(x_n^* - n) \prod_{k=1}^{n} (x_n^* - k)$. Therefore, we have:

$$\Psi(x_n^*) = \sum_{k=1}^{n} \frac{1}{x_n^* - k} + \frac{\alpha}{|\Gamma(x_n^* - n)|}$$

or:

$$f_n(x_n^*) = \frac{\alpha}{|\Gamma(x_n^* - n)|} \tag{6.2}$$

where:

$$f_n(x_n^*) = \Psi(x_n^*) + \sum_{k=1}^{n} \frac{1}{k - x_n^*} \tag{6.3}$$

Note that in the equation (6.2) our left hand side is increasing on the interval $(x_n, 1)$ and our right hand side is decreasing on the interval $(x_n, 1)$. Now let's look at equivalent equation we need to solve to find a solution, on the next interval, to our original equation $|\Gamma'(x)| = \alpha$. That is:

$$f_{n+1}(x) = \frac{\alpha}{|\Gamma(x - n - 1)|}$$

Letting $x = x_n^*$ makes our left side become:

$$\begin{aligned} f_{n+1}(x_n^*) &= \Psi(x_n^*) + \sum_{k=1}^{n} \frac{1}{k - x_n^*} + \frac{1}{n + 1 - x_n^*} \\ &= \frac{\alpha}{|\Gamma(x_n^* - n)|} + \frac{1}{n + 1 - x_n^*} \end{aligned}$$

On the other hand, our right side would be:

$$\frac{\alpha}{|\Gamma(x_n^* - n - 1)|} = \frac{\alpha(n + 1 - x_n^*)}{|\Gamma(x_n^* - n)|}$$

For finding x_{n+1}^*, if we start at the critical point of this interval, x_{n+1}, we will have our left side is zero and our right side is positive. This means that at any point, if we have the value of the left side still smaller than the right side, we know that our solution, x^*_{n+1}, is greater than that point. In other words, if we have:

$$\frac{\alpha}{|\Gamma(x^*_n - n)|} + \frac{1}{n+1-x^*_n} \leq \frac{\alpha(n+1-x^*_n)}{|\Gamma(x^*_n - n)|}$$

then we can conclude that $x^*_{n+1} \geq x^*_n$. The inequality gives us:

$$\begin{aligned}
\frac{1}{n+1-x^*_n} &\leq \frac{\alpha(n-x^*_n)}{|\Gamma(x^*_n - n)|} \\
\alpha &\geq \frac{|\Gamma(x^*_n - n)|}{(n-x^*_n)(n+1-x^*_n)} \\
&= \frac{|\Gamma(x^*_n - 1)|}{(n-x^*_n)(n+1-x^*_n)\prod_{k=2}^{n}(k-x^*_n)} \\
&= \frac{|\Gamma(-y_n)|}{(n-1+y_n)(n+y_n)\prod_{k=1}^{n-1}(y_n+k)} \\
&= \frac{|\Gamma(1-y_n)|}{(n-1+y_n)(n+y_n)\prod_{k=0}^{n-1}(y_n+k)}
\end{aligned}$$

We have:

$$\begin{aligned}
\prod_{k=0}^{n-1}(y_n+k) &= y_n\prod_{k=1}^{n-1}(y_n+k) \\
\prod_{k=0}^{n-1}(y_n+k) &= \left(y_n^2(n-1)!\right)\cdot\left(\frac{\prod_{k=1}^{n}(y_n+k)}{n!}\right)\cdot\left(\frac{n}{y_n(y_n+n)}\right) \\
\lim_{n\to\infty}\prod_{k=0}^{n-1}(y_n+k) &= \frac{1}{\alpha}\cdot 1\cdot\infty = \infty
\end{aligned}$$

Therefore, the limit of the right side of our inequality when n goes to infinity is zero. This means that with n is large enough, our inequality is qualified, since α is positive. In other words, when n is sufficiently large, our sequence $\{x^*_n\}$ is strictly increasing.

7. Properties involving the gamma function

Proposition 7.1.

$$\int_0^\infty e^{-x}\log^2 x dx = \frac{\pi^2}{6} + \gamma^2$$

Proof. From the definition of the gamma function, we can come up with the formula for the second derivative of the gamma function, and that is:

$$\Gamma''(x) = \int_0^\infty t^{x-1} e^{-t} \log^2 t dt$$

Therefore:

$$\Gamma''(1) = \int_0^\infty e^{-t} \log^2 t dt = \int_0^\infty e^{-x} \log^2 x dx$$

To calculate $\Gamma''(1)$, we first need to calculate $\Psi'(1)$. We have the following formula for $\Psi(x)$:

$$\begin{aligned}
\Psi(x+1) &= -\gamma + \sum_{r=1}^\infty \frac{x}{r(r+x)} \\
\Psi'(x+1) &= \sum_{r=1}^\infty \frac{1}{(r+x)^2} \\
\Psi'(1) &= \sum_{r=1}^\infty \frac{1}{r^2} = \zeta(2) = \frac{\pi^2}{6}
\end{aligned}$$

Remember that: $\Psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$. Therefore:

$$\begin{aligned}
\Psi'(x) &= \frac{\Gamma''(x)\Gamma(x) - (\Gamma'(x))^2}{\Gamma^2(x)} \\
\Psi'(1) &= \Gamma''(1) - \gamma^2 \\
\Gamma''(1) &= \frac{\pi^2}{6} + \gamma^2
\end{aligned}$$

Therefore, we have:

$$\int_0^\infty e^{-x} \log^2 x dx = \Gamma''(1) = \frac{\pi^2}{6} + \gamma^2$$

This proves our proposition. □

Proposition 7.2.

$$\lim_{n\to\infty} \left(n - \Gamma\left(\frac{1}{n}\right) \right) = \gamma \tag{7.1}$$

Proof. We have:

$$\begin{aligned}
\lim_{n\to\infty}\left(n-\Gamma\left(\frac{1}{n}\right)\right) &= -\lim_{n\to\infty}\left(\Gamma\left(\frac{1}{n}\right)-n\right)\\
&= -\lim_{n\to\infty}\left(\frac{\Gamma\left(1+\frac{1}{n}\right)}{\frac{1}{n}}-\frac{1}{\frac{1}{n}}\right)\\
&= -\lim_{n\to\infty}\frac{\Gamma\left(1+\frac{1}{n}\right)-1}{\frac{1}{n}}\\
&= -\lim_{n\to\infty}\frac{\Gamma\left(1+\frac{1}{n}\right)-\Gamma(1)}{\frac{1}{n}}\\
&= -\Gamma'(1)=\gamma
\end{aligned}$$

Our proposition is proved. □

In this coming section, I hope you will forgive me the mildest of abuses of notation. When integrating up to or over the discontinous points of $\Gamma(x)$ the proper notation of the underlying limiting process may be omitted for the sake of sanity and simplicity.

8. Area Under the Curve

When considering $\int_a^b \Gamma(x)dx$ several questions arise.

- For what values of a, b does this integral converge?
- For what values of a, b, if any, does this integral diverge?
- As $a \to -\infty$ does this integral converge or diverge?

In order to better understand the answers to these questions, it becomes necessary to first examine the behavior of $\Gamma(x)$ near its points of discontinuity.

Proposition 8.1. For all natural numbers, k, and for real x, $\Gamma(x) \approx \frac{(-1)^k}{x(k!)}$ as $x \to -k$. Formally, we claim $\left|\lim_{x\to 0}\left(\Gamma(x-k)-\frac{(-1)^k}{x(k!)}\right)\right| < \infty$.

Proof. We shall prove this by induction. We first show that $\left|\lim_{x\to 0}\left(\Gamma(x)-\frac{1}{x}\right)\right| < \infty$. We recall the recurrence relation,

$$\Gamma(x) = \frac{\Gamma(x+1)}{x}$$

$$\Rightarrow \lim_{x\to 0}\left(\Gamma(x)-\frac{1}{x}\right) = \lim_{x\to 0}\left(\frac{\Gamma(x+1)}{x}-\frac{1}{x}\right)$$
$$= \lim_{x\to 0}\left(\frac{\Gamma(1+x)-\Gamma(1)}{x}\right)$$
$$= \Gamma'(1)$$
$$= -\gamma$$

We assume that $\left|\lim_{x\to 0}\left(\Gamma(x-n)-\frac{(-1)^n}{x(n!)}\right)\right| < \infty$. To complete our induction we must show $\left|\lim_{x\to 0}\left(\Gamma(x-(n+1))-\frac{(-1)^{(n+1)}}{x((n+1)!)}\right)\right| < \infty$. Recalling the recurrence relation of $\Gamma(x)$, we may write $\Gamma(x-(n+1)) = \frac{\Gamma(x-n)}{x-(n+1)}$.

$$\Rightarrow \lim_{x\to 0}\left(\Gamma(x-(n+1))-\frac{(-1)^{(n+1)}}{x((n+1)!)}\right) = \lim_{x\to 0}\left(\frac{\Gamma(x-n)}{x-(n+1)}-\frac{(-1)^{(n+1)}}{x((n+1)!)}\right)$$
$$= \lim_{x\to 0}\left(\frac{-1}{n+1}\right)\left(\frac{\Gamma(x-n)}{1-\frac{x}{n+1}}-\frac{-1^n}{x(n!)}\right)$$
$$= \lim_{x\to 0}\left(\frac{-1}{n+1}\right)\left(\Gamma(x-n)-\frac{(-1)^n}{x(n!)}\right)$$
$$+ \lim_{x\to 0}\left(\frac{-1}{n+1}\right)\left(\frac{\Gamma(x-n)}{1-\frac{x}{n+1}}-\Gamma(x-n)\right)$$

Since by our inductive hypothesis $\left|\lim_{x\to 0}\left(\Gamma(x-n)-\frac{(-1)^n}{x(n!)}\right)\right| < \infty$, we must now show $\left|\lim_{x\to 0}\left(\frac{\Gamma(x-n)}{1-\frac{x}{n+1}}-\Gamma(x-n)\right)\right| < \infty$.

$$\begin{aligned}
\lim_{x\to 0}\left(\frac{\Gamma(x-n)}{1-\frac{x}{n+1}}-\Gamma(x-n)\right) &= \lim_{x\to 0}\left(\frac{\frac{x\Gamma(x-n)}{n+1}}{1-\frac{x}{n+1}}\right)\\
&= \lim_{x\to 0}\left(\frac{x\Gamma(x-n)}{n+1-x}\right)\\
&= \lim_{x\to 0}\left(\frac{x\left(\frac{(-1)^n}{x(n!)}\right)\left(\frac{\Gamma(x-n)}{\frac{(-1)^n}{x(n!)}}\right)}{n+1-x}\right)\\
&= \lim_{x\to 0}\left(\frac{(-1)^n}{(n!)}\right)\left(\frac{\left(\frac{\Gamma(x-n)}{\frac{(-1)^n}{x(n!)}}\right)}{n+1-x}\right)
\end{aligned}$$

Now since we have $\left|\lim_{x\to 0}\left(\Gamma(x-n)-\frac{(-1)^n}{x(n!)}\right)\right| < \infty$, and since each term individually diverges to $\pm\infty$, it immediately follows that $\lim_{x\to 0}\left(\frac{\Gamma(x-n)}{\frac{(-1)^n}{x(n!)}}\right)\to 1$. So we may conclude that

$$\lim_{x\to 0}\left(\frac{(-1)^n}{(n!)}\right)\left(\frac{\left(\frac{\Gamma(x-n)}{\frac{(-1)^n}{x(n!)}}\right)}{n+1-x}\right) = \frac{(-1)^n}{(n+1)!}$$

And so, since this limit is finite as well, we have $\left|\lim_{x\to 0}\left(\Gamma(x-(n+1))-\frac{(-1)^{(n+1)}}{x((n+1)!)}\right)\right| < \infty$, completing our induction. In point of fact, our method of solution not only implies that this limit is finite, but provides a method of calculating it. If we denote the value of this limit by $C_n = \lim_{x\to 0}\left(\Gamma(x-(n+1))-\frac{(-1)^{(n+1)}}{x((n+1)!)}\right)$, we may establish a recurrence relation,

$$\begin{aligned}
C_0 &= -\gamma\\
C_n &= \frac{-1}{n}\left(C_{n-1}\right)+\frac{(-1)^n}{(n)(n!)}
\end{aligned}$$

□

Armed with this proof, we are ready to answer some of our original questions about the Gamma function.

Proposition 8.2. $\int_{b-\delta}^{b} \Gamma(x)dx$ diverges if b is either zero or a negative integer, and for any arbitrary $\delta \in (0,1)$.

Proof. Before we begin, we shall make a definition which shall be used throughout the rest of this paper. We define,

$$E_k(x) = \Gamma(x-k) - \frac{(-1)^k}{x(k!)}$$

We are now well-equipped to tackle the proof of our proposition.

$$\begin{aligned}\int_{b-\delta}^{b} \Gamma(x)dx &= \int_{-\delta}^{0} \Gamma(x-b)dx \\ &= \int_{-\delta}^{0} \frac{(-1)^b}{x(b!)} + \int_{-\delta}^{0} E_b(x)dx\end{aligned}$$

$\int_{-\delta}^{0} \frac{(-1)^b}{x(b!)}$ clearly diverges, so all we need to demonstrate is the convergance of $\int_{-\delta}^{0} E_b(x)dx$ and we will have shown that $\int_{-\delta}^{0} \Gamma(x-b)dx$ diverges.

Certainly for all $x \in (0,1)$ $E_b(x)$ is finite, and by Proposition 8.1 $\lim_{x\to 0} E_b(x)$ is finite. Now, since $E_b(x)$ is continuous, we may conclude that $E_b(x)$ is bounded on the interval $(-\delta, 0)$. Since we are integrating a function over a finite interval upon which it is bounded we may conclude that,

$$\left| \int_{-\delta}^{0} E_b(x)dx \right| < \infty$$

$$\Rightarrow \int_{b-\delta}^{b} \Gamma(x)dx \to \pm\infty$$

□

It is trivial to show that a nigh-identical proof could be used to show that $\int_{a}^{a+\delta} \Gamma(x)dx$ diverges as well if a is either 0 or a negative integer and $\delta \in (0,1)$.

Proposition 8.3. Although $\int_{k-\delta}^{k+\delta} \Gamma(x)dx$ is not well-defined for all negative integers, k and $\delta \in (0,1)$, the Cauchy principal value defined by $\int_{0}^{\delta} (\Gamma(x-k) + \Gamma(-x-k))\, dx$ is well-defined and finite.

Before we begin the proof it is important to stress that we may only take a principal value of this integral. $\int_{k-\delta}^{k} \Gamma(x)dx \to \pm\infty$ and $\int_{k}^{k+\delta} \Gamma(x)dx \to \mp\infty$, so the value of this integral is not well-defined unless we consider its principal value.

Proof.

$$\begin{aligned}\int_0^\delta (\Gamma(x-k)+\Gamma(-x-k))\,dx &= \int_0^\delta \left(\frac{(-1)^k}{x(k!)}+\frac{(-1)^k}{(-x)(k!)}+E_k(x)+E_k(-x)\right)dx \\ &= \int_0^\delta (0+E_k(x)+E_k(-x))\,dx \\ &= \int_{-\delta}^\delta E_k(x)dx\end{aligned}$$

As before, for all $x \in (0,1)$ and for all $x \in (-1,0)$ $E_k(x)$ is finite, and by Proposition 8.1 $\lim_{x\to 0} E_k(x)$ is finite. Now, since $E_k(x)$ is continuous, we may conclude that $E_k(x)$ is bounded on the interval $(-\delta,\delta)$. Since we are integrating a function over a finite interval upon which it is bounded we may conclude that,

$$\left|\int_{-\delta}^\delta E_k(x)dx\right| < \infty$$

$$\Rightarrow \left|\int_0^\delta (\Gamma(x-k)+\Gamma(-x-k))\,dx\right| < \infty$$

□

Proposition 8.4. $\int_a^b \Gamma(x)dx$ is finite if neither of a,b are 0 or a negative integer, provided we take the value of the integral over any point of discontinuity to be the Cauchy principal value thereof.

Proof. This is an immediate consequence of proposition 8.3. Let n denote $\lfloor b-a \rfloor$ We may write

$$\int_a^b \Gamma(x)dx = \sum_{k=1}^n \left(\int_{a+k-1}^{a+k} \Gamma(x)dx\right) + \int_{a+n}^b \Gamma(x)dx$$

By proposition 8.3 we know that the principal value of each of these integrals is finite. So, we have a finite sum of finite terms, which therefore must be finite. □

Proposition 8.5. $\int_a^b \Gamma(x)dx$ diverges if at least one of a,b is either 0 or a negative integer.

Proof. If only one of a,b is either 0 or a negative integer, then this proof immediately follows from proposition 8.4. Without loss of generality assume that b is either 0 or a negative integer. Further let us assume that $b-a>1$ since if it is not, then this is true by proposition 8.2

$$\int_a^b \Gamma(x)dx = \int_a^{b-\delta} \Gamma(x)dx + \int_{b-\delta}^b \Gamma(x)dx$$

We know that $\int_a^{b-\delta} \Gamma(x)dx$ is finite by proposition 8.4 and we know that $\int_{b-\delta}^b \Gamma(x)dx$ diverges by proposition 8.2. So, $\int_a^b \Gamma(x)dx$ diverges as well.

Now let us consider the case where both a, b are one of either 0 or a negative integer. We need to show that we cannot take a principal value in this case. Let $n = b - a$.

$$\begin{aligned}\int_a^b \Gamma(x)dx &= \int_a^{a+n} \Gamma(x)dx \\ &= \int_a^{a+\frac{1}{2}} \Gamma(x)dx + \int_{a+\frac{1}{2}}^{a+n-\frac{1}{2}} \Gamma(x)dx + \int_{a+n-\frac{1}{2}}^{a+n} \Gamma(x)dx\end{aligned}$$

We know $\int_{a+\frac{1}{2}}^{a+n-\frac{1}{2}} \Gamma(x)dx$ to be finite by proposition 8.4. So we must now show that we cannot take a principal value for $\int_a^{a+\frac{1}{2}} \Gamma(x)dx + \int_{a+n-\frac{1}{2}}^{a+n} \Gamma(x)dx$.

$$\begin{aligned}\int_a^{a+\frac{1}{2}} \Gamma(x)dx + \int_{a+n-\frac{1}{2}}^{a+n} \Gamma(x)dx &= \int_0^{\frac{1}{2}} \Gamma(x-a)dx + \int_{-\frac{1}{2}}^{0} \Gamma(x-(a+n))dx \\ &= \int_0^{\frac{1}{2}} \Gamma(x-a)dx + \int_0^{\frac{1}{2}} \Gamma(-x-(a+n))dx \\ &= \int_0^{\frac{1}{2}} (\Gamma(x-a) + \Gamma(-x-(a+n))dx \\ &= \int_0^{\frac{1}{2}} \frac{(-1)^a}{x(a!)} + \frac{(-1)^{a+n}}{x((a+n)!)} + E_a(x) + E_{a+n}(x)dx\end{aligned}$$

Now, since $\int_0^{\frac{1}{2}} E_a(x)dx$ and $\int_0^{\frac{1}{2}} E_{a+n}(x)dx$ are both finite, we must have that $\int_a^b \Gamma(x)dx$ converges if and only if $\int_0^{\frac{1}{2}} \frac{(-1)^a}{x(a!)} + \frac{(-1)^{a+n}}{x((a+n)!)}dx$ does. It can be clearly seen that this only happens when $n = 0$, so we are done. □

Proposition 8.6. $\int_{-\infty}^b \Gamma(x)dx$ does not converge for any value of b.

Proof. We will prove this by contradiction. We assume that there exists a real number c such that

$$\int_{-\infty}^{b} \Gamma(x)dx = c$$

This implies that $\forall \epsilon > 0 \ \exists$ a real number N such that $\forall n > N$ $\left| \left(\int_{-n}^{b} \Gamma(x)dx \right) - c \right| < \epsilon$. However, all we have to do is choose an integer $n > N$ and $\int_{-n}^{b} \Gamma(x)dx$ diverges, therefore this cannot be true. $\square$