SIEVE THEORY AND ITS APPLICATION TO THE FIBONACCI SERIES

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Abstract. It is unknown whether or not there are an infinite number of primes in the Fibonacci sequence, although there has been much study on the topic. For example, it has been shown that $F_n|F_{an}$ for any natural number a. Another important discovery is that the Fibonacci sequence is periodic for a modulus of any base. Moreover, Marc Renault has published an in-depth study of the behavior of the Fibonacci sequence in modular arithmetic, and demonstrated that the sequence has exactly one, two, or four zeros per period regardless of which base is chosen; it has also been shown that each term in the Fibonacci sequence has a prime factor that has not previously shown up in the sequence. Despite these discoveries, the question of whether the Fibonacci sequence contains an infinite number of primes remains unanswered. Our project sets out to explore this in greater depth by using sieve methods, specifically the Brun sieve. These sieve ideas will be extended to a probabilistic approach, and we conclude with a linear algebraic approach using matrices built from sieved sets.

1. Sieve Theory

1.1. **The Sieve of Eratosthenes.** A sieve is a method to count or estimate the size of "sifted sets" of integers. These sets consist of the numbers that remain after the rest have been "filtered" out. The classic example is the sieve of Eratosthenes, an algorithm for finding all prime numbers less than a given integer. This method only works for small integers. It was thought to have been created by an ancient Greek, Eratosthenes, who lived around 250 B.C. The basic concept behind his sieve is to filter the prime numbers out of a finite list of consecutive integers. Eratosthenes' sieve does this by taking the list from 2 to n, and then letting the initial prime number, denoted p_1, be equivalent to the first prime number, 2. Then, using p_1, strike every multiple of p_1 from the finite list of consecutive integers. Now take the next number on the list that has not been struck through. This is the next prime number, denoted p_2. Repeat the process listed above. In the Eratosthenes sieve, repeat this process until p_k^2 is greater than

n. Every number that has not been struck off that is left is a prime number.

The remarks above gives us this expression of numbers: $\pi(n)-\pi(n^{\frac{1}{2}})$ of primes denoted p, satisfying $n^{\frac{1}{2}} < p \leq n$. In this case, $\pi(x)$ is the prime counting function, which is the function that counts the number of primes less than or equal to some real number x. His principle can also be expressed by:

$$1 + \pi(n) - \pi(n^{\frac{1}{2}}) = \sum_{a \leq n} s^{(0)}(a).$$

In this expression, the "sifting function" is defined by

$$s^{(0)}(a) = \begin{cases} 1 & \text{if a is not divisible by any prime } p \leq n^{\frac{1}{2}}, \\ 0 & \text{if a is divisible by some prime } p \leq n^{\frac{1}{2}}. \end{cases}$$

As was previously stated, the sieve of Eratosthenes works well for smaller numbers, but as the sets grow larger so does the error terms for the estimates. Modern mathematicians have tried to combat this occurence by developing more refined sieve methods. One of these methods is the Brun sieve, which ahs been shown to be useful when focused on certain intractible problems in prime number theory.

1.2. **Brun's Sieve.** Viggo Brun was a twentieth century, Norwegian mathematician. He lived from 1885-1978 and attended the University of Oslo, where he eventually settled as a professor. He created the Brun sieve method based on the sieve of Eratosthenes to use on additive problems in 1915. He used this method to prove that there exists infinitely many integers n such that n and $n+2$ have at most nine prime factors, and that all large, even integers are the sum of two "nine-almost primes." A k-almost prime is a natural number that has exactly k prime factors, counted with multiplicity. A number n is k-almost prime if and only if $\Omega(n) = k$, where $\Omega(n)$ is the number of primes in the prime factorization of n. A number is prime in the usual sense of the word if it is 1-almost prime.

The Brun sieve is a method for estimating the size of a sifted set of positive integers that satisfy conditions expressed by congruence. Congruence refers to an equivalence relation that is based on an algebraic structure that is compatible with the structure. This sieve is a combinatorial type, meaning it is derived from use of the inclusion-exclusion principle. This principle states that if A and B are two finite sets, then the following is true:

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

We must let $A = \{x \in \mathbb{N} \mid x \leq n, \text{ some } n\}$ and let $\mathcal{P}$ be a set of primes. For all $p \in \mathcal{P}$, we let A_p represent the set of elements in A properly divisible by p, more formally $A_p = \{x \in A \mid x = kp, k \neq 1\}$. For $d = p_1 \cdots p_n$, we extend this to let A_d the intersection of the A_p for p dividing d, when d is a product of distinct primes from $\mathcal{P}$. To be more rigorous, , let $A_d = \cap_{i=1}^{n} A_{p_i}$. It is clear that only a finite number of primes are needed to sieve a finite set of integers. Suppose z is the largest prime needed, then let $P(z)$ be the set of all primes less than or equal to z. Then we can formally state

$$S(A, \mathcal{P}, z) = |A \backslash \bigcup_{p \in P(z)} A_p|$$

We recall that $|A_d|$ may be thought of as a divisor function, i.e. a function of the prime divisors of d. Because divisor functions are multiplicative, we define $w(d)$ to be a multiplicative function such that

$$|A_d| = \frac{w(d)}{d} X + R_d$$

where $X = |A|$ and R_d is a remainder.We may use this simple example to demonstrate the Brun sieve. Let us take

$$A = \{2, 5, 6, 7, 10, 11, 12, 13, 18, 20, 22, 24, 28, 35\}$$

and

$$\mathcal{P} = \{2, 4, 10\}.$$

Then we can see, by sifting the given A with the given $\mathcal{P}$ that

$$A_2 = \{6, 10, 12, 18, 20, 22, 24, 28\},$$

$$A_4 = \{12, 20, 24, 28\},$$

$$A_{10} = \{20\}.$$

The following estimates and formulations concerning the Brun sieve can be found in [3], chapter 6.

Theorem 1.1. *(Theorem 6.12, [3])We keep the above notation and have 3 further hypotheses:*

1. *$|R_d| \leq w(d)$ for all squarefree d composed of elements of $P(z)$*
2. *there exists a positive constant C such that $w(p) < C$ for any $p \in P(z)$*
3. *there exists positive constants C_1, C_2 such that*

$$\sum_{p \in P(z)} \frac{w(p)}{p} < C_1 \log \log z + C_2$$

Then $S(A, P, z) = XW(z)(1 + O((\log z)^{-A})) + O(z^{\eta \log\log z})$ *with* $A = \eta \log \eta$ *and* $W(z) = \Pi_{p|P(z)}(1 - \frac{w(p)}{p})$. *In particular, if* $\log z \leq c\frac{\log x}{\log\log x}$ *for a suitable positive constant c that is sufficiently small, then we have* $S(A, \mathcal{P}, z) = XW(z)(1 + o(1))$.

2. The Fibonacci Sequence

The Fibonacci Sequence is a sequence of numbers, F_n, that are defined by the reoccurring relation:

$$F_n = F_{n-1} + F_{n-2}.$$

They have seed values of $F_0 = 0$ and $F_1 = 1$. The Fibonacci sequence is found and used in several fields, from finances to nature.

We will look at specific Fibonacci numbers, the Finbonacci primes. As the name suggests, a Fibonacci prime is a Fibonacci number that is prime. It is unknown if there is a finite or infinite number of Fibonacci primes. Fibonacci numbers that have a prime index of p have no common divisors greater than or equal to one with its preceding Fibonacci numbers because of the following identity:

$$GCD(F_p, F_m) = F_{GCD(n,m)},$$

for $n \geq 3$,with F_n dividing F_m if and only if n divides m. If we let m be a prime number p that is greater than n, then F_p may not share any common divisors with any preceding Fibonacci numbers. This will be shown later in this paper. The American mathematician, Carmichael, gives us his theorem that states for n > 12, the nth Fibonacci number (F_n) has at least one prime factor that is not a factor of any previous Fibonacci number.

(1) Applying the Brun sieve to the Fibonacci sequence.

To use this method, the three conditions set forth in the description of the theorem must first be satisfied.

(a) $|R_d| \leq \omega(d)$ $\forall$ squarefree d composed of primes $\mathcal{P}$. If d is squarefree, then $\omega(d) = \omega(p_1), \omega(p_2), ..., \omega(p_k)$ because ω is multiplicative. Since $\omega(d)$ is approximating a counting function, we can choose it so that this criterion is met.

(b) $\exists C \geq 0$ such that $\omega(p) < C$ $\forall p \in \mathcal{P}$. In the case of the Fibonacci numbers, $|\mathcal{P}| = \lambda(F_n)$ where $F_n = \max\{\mathcal{F}\}$. We created this function to tell us how many primes were needed to fully sieve the first n Fibonacci numbers. We let this function be $\lambda(F_n)$.

We define $\lambda(F_n) = p$, where p is the index of largest prime needed to sieve $\mathcal{F}$ completely. For example, if we wish to sieve the first 8 non-trivial Fibonacci numbers, $\{2, 3, 5, 8, 13, 21, 34, 55\}$, we need to sieve with $\{2, 3, 5, 13\}$, so $\lambda(55) = 6$ since 55 is the

10^{th} Fibonacci number and 13 is the 6^{th} prime number. We give the following table of small values of lambda:

F_n	2	3	5	8	13	21	34	55	89	144	233	377
$\lambda(n)$	1	2	3	3	6	6	6	6	24	24	51	51

(c) $\exists C_1, C_2 \geq 0$ such that

$$\sum_{p<z, p\in\mathcal{P}} \frac{\omega(p)}{p} < C_1 \log\log(z) + C_2.$$

Because we are sieving with a finite list of the Fibonacci numbers, we know that the sum will be finite, so $\exists C_1, C_2$ for which this condition is true.

We may now apply Brun's sieve to the Fibonacci sequence, because we know all of the conditions are satisfied. We recall

$$S(A, \mathcal{P}, z) = |A \setminus \bigcup_{p\in P(z)} A_p|$$

. In the case of the Fibonacci sequence, we must replace A with F_n, the Fibonacci sequence. We also let $\mathcal{P}$ be equivalent to all prime numbers. To truly apply the Brun sieve, we must only take a finite section of the Fibonacci sequence. Our refined sieve may be stated as such:

$$S(F_n, \mathcal{P}, z) = |F_n \setminus \bigcup_{p\in P(z)} F_p|.$$

We take

$$F_n = 2, 3, 5, 8, 13, 21, 34, 55.$$

In this case, to completely sieve it, we must take $\mathcal{P} = \{p \,|\, p \leq \lambda(55), p \text{ prime}\}$. Using this many primes assures that every Fibonacci number in the list will appear in at least one A_p.

$$F_2 = \{8, 34\}$$
$$F_3 = \{21\}$$
$$F_5 = \{55\}$$
$$F_7 = \{21\}$$
$$F_{11} = \{55\}$$
$$F_{13} = \emptyset.$$

We may stop there because every term in F_n has been sifted. This gets more difficult as we progress down the Fibonacci sequence, because it gets very large very rapidly, and so does the number of primes required to sift it.

So we have that

$$S(F_n, \mathcal{P}, z) = |\{2, 3, 5, 13\}| = 4$$

.

3. More Fibonacci Primes

In this section, we will begin to look at the Fibonacci number modulo a prime in order to study prime Fibonacci numbers. Recall that it has been shown in [6] that the Fibonacci sequence is periodic modulo any integer, which clearly includes the primes. We begin with a useful lemma.

Lemma 3.1. *If F_n is the n^{th} term of the Fibonacci sequence, then $(F_{n+1}, F_n) = 1$ for all n.*

Proof. Assume not. Then there exists some pair of adjacent terms share a common factor: $F_{n+1} = kl$; $F_n = km$. Then $F_{n-1} = F_{n+1} - F_n = k(l - m)$. Repeating the argument for F_{n-2}, F_{n-3}, $\ldots$, F_2, we see that $F_1 = kj$ for some j. But $F_1 = 1$ and hence $k = 1$. This contradicts the assumption that $k > 1$ and hence the assumption is false. Thus, $(F_{n+1}, F_n) = 1$ for all n. □

Theorem 3.2. *(Carmichael) For all $n > 12$, there exists a prime p_n such that $F_n \equiv 0$ mod p_n and $F_m \not\equiv 0$ mod p_n for all $m < n$.*

Proof. [2] □

Theorem 3.3. *Let $P = \{P\}$ be an arbitrary finite collection of primes. Then there exists a Fibonacci number that has no factors in P.*

Proof. The Fibonacci sequence is periodic modulo p for any arbitary prime [6]. More specifically, the sequence has a period $q < p^2$ because once the sequence repeats two consecutive terms, the periodicity follows since each number is based only on the last two terms. There are only p choices for a number modulo p in each of these slots, and the sequence $0, 0$ is invalid, since all terms would be 0, which is clearly not the case. Note that as the sequence begins to repeat, the q^{th} term is 0 and and $q + 1^{th}$ term is 1.

Now, suppose that for the list of primes $p_1, p_2, ...p_n$, you have the corresponding lists of periods $q_1, q_2, ...q_n$. Now let $d = \text{lcm}(q_1, \ldots, q_n)$.

The d^{th} term of the Fibonacci sequence is necessarily a multiple of all primes in the list, that is, $F_d \equiv 0$ mod p for all p. Now, the Fibonacci sequence is periodic for all primes in the sequence, and the sequences for all the primes are, at this point, starting over. Thus, $F_{d+1} \equiv 1$ mod p for all p and a Fibonacci number relatively prime to all numbers in the list has been found. □

The prime numbers and the Fibonacci numbers are both infinite sets. We would like to see if their intersection also forms an infinite set, but first we will examine each alone.

For the prime numbers, it is accepted that if $\pi(X)$ is the number of primes less than x, then $\pi(x) \sim \frac{x}{\ln x}$, that is, the number of primes less than or equal to x is asymptotic to $\frac{x}{\ln x}$. From this, we see that the probability that a random integer in the interval $[1, x]$ has a probability

$$P_{prime} = \frac{\frac{x}{\ln x}}{x} = \frac{1}{\ln x}$$

of being prime, for large x.

Similarly, we can find the probability that a random integer in the interval $[1, x]$ appears in the Fibonacci sequence. Using the fact that $\lim_{x\to\infty} \frac{F_{n+1}}{F_n} = \phi$, where $\phi = \frac{1+\sqrt{(5)}}{2}$ denotes the golden ratio, we can find a good approximation of the Fibonacci sequence; specifically, $F_n = c\phi^n$ ($c \approx .4472$). Solving for n we find the number of Fibonacci numbers below an arbitrary number: $n \approx \frac{\ln F_n}{\ln \phi + \ln c}$. Similarly, if we are not given the greatest Fibonacci number in our range but are given a number x, we can use the same approximation since x is naturally on the order of the highest Fibonacci number F_n below it (more specifically, x is no greater than ϕF_n).

Thus, $n \approx \frac{\ln x}{\ln \phi + \ln c}$, or simply $n \approx \frac{\ln x}{\ln \phi}$ for large x. Thus, the probability of selecting a Fibonacci number by randomly choosing an integer in the interval $[1, x]$ is

$$P_{Fibonacci} = \frac{\frac{\ln x}{\ln \phi}}{x}$$

for large x.

Now, it is possible to analyze the intersection of the sets. If the prime numbers and the Fibonacci numbers are independent, we can find the probability that a random integer in the $[1, x]$ range is both prime and Fibonacci by simply multiplying the probabilities:

$$P_{(P\cap F)} = P_{prime} \cdot P_{Fibonacci} = \frac{1}{\ln x} \cdot \frac{\ln x}{x \ln \phi} = \frac{1}{x \ln \phi}$$

This sum clearly diverges, as $\ln \phi$ is simply a constant: $\ln \phi \approx .4812$ and the remaining term, when summed from one to infinity, is simply the harmonic series. Thus, $\sum P_{(P\cap F)} \to \infty$ and we expect there to be an infinite number of primes from this probabilistic argument. However, there is a slight flaw in this argument as choosing an integer in F and choosing an integer in P are not, in fact, independent events probabilistically. For instance, for any composite $n \neq 4$, F_n is not prime. This restriction does not, however, apply to prime n so it might be reasonable to analyze the probability for only prime n instead of all the integers. The sum $\sum_x P_{(P\cap F)} = \frac{1}{x \ln \phi}$ still diverges, because the sum $\sum_p \frac{1}{p}$ still diverges. This is an indication that the number of Fibonacci primes is, in fact, infinite, but since choosing numbers from the Fibonacci and prime sets are still not independent events, this does not constitute a complete proof.

4. The Matrix Approach

We proceed now by creating a matrix that will allow us to easily identify Fibonacci primes. If we create an infinite array with the Fibonacci numbers labeling the columns and the primes labeling the rows, then fill the array by reducing the Fibonacci numbers modulo the the prime numbers. The result is

$$
\begin{array}{c|ccccccccc c}
 & 2 & 3 & 5 & 8 & 13 & 21 & 34 & 55 & 89 & \cdots \\
\hline
2 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & \\
3 & 2 & 0 & 2 & 2 & 1 & 0 & 1 & 1 & 2 & \\
5 & 2 & 3 & 0 & 3 & 3 & 1 & 4 & 0 & 4 & \\
7 & 2 & 3 & 5 & 1 & 6 & 0 & 6 & 6 & 5 & \\
11 & 2 & 3 & 5 & 8 & 2 & 10 & 1 & 0 & 1 & \\
13 & 2 & 3 & 5 & 8 & 0 & 8 & 8 & 3 & 11 & \\
17 & 2 & 3 & 5 & 8 & 13 & 4 & 0 & 4 & 4 & \\
19 & 2 & 3 & 5 & 8 & 13 & 2 & 15 & 17 & 13 & \\
23 & 2 & 3 & 5 & 8 & 13 & 21 & 11 & 9 & 20 & \\
\vdots & & & & & & & & & & \ddots
\end{array}
$$

The reader may notice that once a Fibonacci number appears in a column, every number below that entry is the same Fibonacci number. With this in mind, no information is lost if we ignore all the entries below the first appearance of a Fibonacci number, and the first appearance of the next Fibonacci number in the column immediately to the right. Algorithmically, one can enumerate the primes $p_1, \ldots,$ and compare the n^{th} prime with the $F(n-2)$ (the index on the Fibonacci number has to be shifted since the array ignores $F_1 = F_2 = 1$). When the first n is found such that $p_n < F_{n-2}$, that row is deleted. Now reenumerate the remaining primes, and repeat the first step. Finally, one can simply consider the array as a matrix, as we do here.

When this sieving process is completed, we have the following matrix, with the labels still shown for clarity.

$$\begin{array}{c|ccccccccc c}
 & 2 & 3 & 5 & 8 & 13 & 21 & 34 & 55 & 89 & \cdots \\
\hline
2 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & \\
3 & 2 & 0 & 2 & 2 & 1 & 0 & 1 & 1 & 2 & \\
5 & 2 & 3 & 0 & 3 & 3 & 1 & 4 & 0 & 4 & \\
11 & 2 & 3 & 5 & 8 & 2 & 10 & 1 & 0 & 1 & \\
13 & 2 & 3 & 5 & 8 & 0 & 8 & 8 & 3 & 11 & \\
29 & 2 & 3 & 5 & 8 & 13 & 21 & 5 & 26 & 2 & \\
37 & 2 & 3 & 5 & 8 & 13 & 21 & 34 & 18 & 15 & \\
59 & 2 & 3 & 5 & 8 & 13 & 21 & 34 & 55 & 30 & \\
89 & 2 & 3 & 5 & 8 & 13 & 21 & 34 & 55 & 0 & \\
\vdots & & & & & & & & & & \ddots
\end{array}$$

Notice, by reducing the array as we did, all the prime Fibonacci numbers are represented by a zero on the diagonal. Every nonzero main diagonal entry is the Fibonacci number indexed by the column label. Furthermore, every subdiagonal is exactly the Fibonacci sequence.
To use any linear algebraic tools on these matrices, we must consider only finite submatrices.

Definition 4.1. Let AN be the $N \times N$ submatrix of the above infinite matrix, that consists of the first N rows and first N columns. For notation purposes, let RN_k denote the k^{th} row of AN and let $AN_{(i,j)}$ denote the $(i,j)^{th}$ position of AN. In the case of varying N, parantheses will be used. For instance, $A(N+1)$ is the similarly defined $(N+1) \times (N+1)$ matrix.

Example 4.2. We give a few of the matrices here, with their determinants and ranks.

$$A3 = \begin{pmatrix} 0 & 1 & 1 \\ 2 & 0 & 2 \\ 2 & 3 & 0 \end{pmatrix}$$

$\det(A3) = 10$ $\text{rk}(A3) = 3$

$$A4 = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 2 & 0 & 2 & 2 \\ 2 & 3 & 0 & 3 \\ 2 & 3 & 5 & 8 \end{pmatrix}$$

$\det(A4) = 60$ $\text{rk}(A4) = 4$

$$A7 = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 2 & 0 & 2 & 2 & 1 & 0 & 1 \\ 2 & 3 & 0 & 3 & 3 & 1 & 4 \\ 2 & 3 & 0 & 3 & 3 & 1 & 0 \\ 2 & 3 & 5 & 8 & 2 & 10 & 1 \\ 2 & 3 & 5 & 8 & 0 & 8 & 8 \\ 2 & 3 & 5 & 8 & 13 & 21 & 5 \\ 2 & 3 & 5 & 8 & 13 & 21 & 34 \end{pmatrix}$$

$\det(A7) = 0$ $\text{rk}(A7) = 6$

Theorem 4.3. *For all* $n \geq 6$, $\det(AN) = 0$.

Proof. We proceed by induction on N. Let $N = 6$, then a few moments with $A6$ will yield

$$R6_6 = R6_5 + \frac{13}{2}(R6_4 - R6_5)$$

Hence the rows are not linearly independent and therefore $\det(A6) = 0$.

Assume $\det(AN) = 0$ for all $N/leqK$. Then we have $AK_k = \sum_{i=1}^{K-1} c_i RK_i$ for some constants c_i. Consider now the linearly combination of rows

$$(R(K+1)_k - \sum_{i=1}^{K-1} c_i R(K+1)_i) \cdot \frac{b}{a} + R(K+1)_k$$

where $a = \sum_{i=1}^{K-1} c_i A(K+1)_{(i,K+1)}$ and $b = A(K+1)_{(K+1,K+1)} - \sum_{i=1}^{K-1} c_i A(K+1)_{(i,K+1)}$.

We see that this linear combination trivially gives the first K entries of $R(K+1)$ by the construction of AN. Namely, that each row is exactly equal to the previous row under the main diagonal. Hence, we only need to check what the combination gives us in the last position. To simplify notation, let $P = A(K+1)_{(K,K+1)}$, $Q = A(K+1)_{(K+1,K+1)}$ and $R = \sum_{i=1}^{K-1} c_i A(K+1)_{(i,K+1)}$. Then the linearly combination above, applied to the $(k+1)^{st}$ column is

$$(P-R)\cdot\frac{Q-R}{P-R}+R$$

This easily simplifies to Q, which is $A(K+1)_{(K+1,K+1)}$, exactly the value we desire and in the correct position. Hence we have that the $(K+1)^{st}$ row is a linear combination of the previous rows, and thus, $\det(A(K+1))=0$ as required. □

Since one row in AN where $N \leq 6$ is a linear combination of the others, a viable conjucture would be $rank(AN) \leq N-1$.

Theorem 4.4. *For all $N > 3$, we have that $RN_N - RN_{N-1} = (0,0,...0,a,b)$. Furthermore, $a \neq 0$ if and only if either F_N or F_{N-1} are prime.*

Proof. We have a couple of cases to consider that will not be formally separated as the setup is the same for each case. First suppose the the Fibonacci number labeling the column, F_j, is not prime. Then the diagonal entry associated with that row is nonzero, and is in fact exactly the Fibonacci number labeling the column. Furthermore, by construction, $AN_{(j,j-1)} = F_{j-1}$. This clearly holds true for the N^{th} column of AN if F_N isn't prime, so this is where we focus our attention. Again by construction, $AN_{(N,N)} = F_N$, and $AN_{(N,N-1)} = F_{N-1}$. If F_{N-1} is not prime, then $AN_{(N-1,N-1)} = F_{N-1}$ and hence $RN_k - RN_{k-1} = (0,0,...0,a,b)$ where $a=0$. If either F_N or F_{N-1} are prime, then that diagonal entry is 0, and hence $a \neq 0$, thus completing the proof. □

We note the following important fact that is the driving force behind the development of AN. If the Fibonacci primes are finite, there exists an N such that all the Fibonacci primes are less than or equal to F_N. Hence for all $K > N$,

$$AK = \begin{pmatrix} AN & * \\ * & B \end{pmatrix}$$

Where $\text{tr}B = \sum_{i=N+1}^{K} F_i$. I.e., the matrix B has no zeros on the main diagonal, since all the Fibonacci primes occured prior to FN. The goal is that one could use the fact that all the determinants are 0 after $N=5$, coupled together with the finite number of zeros on the main diagonal, and the number of zeros occuring in each row as mentioned in the introduction (1,2, or 4 zeros per period), to reach a possible contradiction to the finiteness of Fibonacci primes.

We end with a conjecture that may help in the aforementioned desired contradiction. Experimental data has shown the following.

Conjecture 4.5. *For all* $N \geq 6$, $rk(AN) = N - 1$.

5. Conclusion

We have shown that Brun's sieve can be applied to the Fibonacci sequence to get estimates of the density of prime Fibonacci numbers. We have also taken a probabilistic approach that, with some refining, could lead to either a probability of 0 of a large Fibonacci number being prime, or a nonzero probability. Either conclusion would be extremely meaningful, and would likely lead to even more questions. Finally, in the matrix approach, we developed the framework one might use to find a contradiction in the assumption that Fibonacci primes are finite. If they are finite, we have a matrix of the form

$$AK = \begin{pmatrix} AN & * \\ * & B \end{pmatrix}$$

where B has no zeros on the main diagonal, but $\det(AK) = 0$. The trick could be wrapped up in matrix partitions, as someone may discover.

Since the Fibonacci numbers have been studied in great detail, and the prime numbers in even greater detail, it is safe to say that any problem involving the intersection of the two sets that has remained unsolved into modern times is a difficult one. We hope to have shed a little light on the problem, and maybe to have developed some novel new ways of approaching it.

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