REALIZING ZERO DIVISOR GRAPHS

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Abstract. In understanding the underlying structures of a commutative ring with unity, it helps to find illustrative ways of seeing the relationship of its elements. Zero divisor graphs is an overlap of Ring Theory and Graph Theory which specifically illustrates the relationships between a ring's zero divisors. It is simple to construct these zero divisor graphs given a commutative ring with unity, but the opposite endeavor is more complicated, i.e. supply a commutative ring for the given graphs.

1. Zero Divisors Graphs

1.1. **Zero Divisors and Their Graphs.** It is important to state that we only considered rings with the additional properties that multiplication is commutative and has unity. Now we can jump right into zero divisors

Definition 1.1. A **zero divisor** is a element r in a ring R, such that $r \cdot s = 0$ for some non-zero s in R.

Definition 1.2. A **zero-divisor graph** is a graph of a ring R where vertices r and s are connected such that $r \cdot s = 0$. There are three types of zero-divisor graphs:

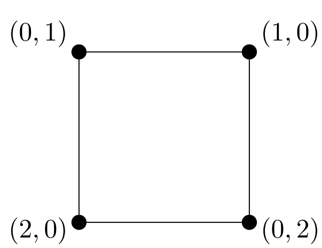
- Beck (1980)
- Anderson-Livingston (1999)
- Mulay (2002)

The *Beck* graph designats a vertex for each zero divisor of the ring, but the structure of these graphs are not very interesting since all elements would be adjacent to the element 0. We can see more interesting structure if we were to eliminate the zero element from the graphs. These are the graphs that *Anderson and Livingston* focused on. Therefore, when we consider a graph for being a possible zero divisor graph, we treated the graph as if it is and *Anderson and Livingston* zero divisor graph.

Though our project called for us to find rings associated with given graphs, where possible, but looking at how these graph are constructed will help understand the relationships illustrated in a zero divisor graph.

Example 1.3. Consider the ring $\mathbb{Z}_3 \times \mathbb{Z}_3$

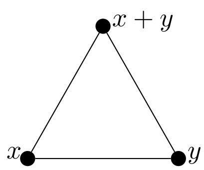
The elements of this ring are: $(0,0)$, $(1,0)$, $(0,1)$, $(1,1)$, $(2,0)$, $(0,2)$, $(2,2)$, $(2,1)$, $(1,2)$



$$(0,1) \cdot (1,0) = (0,0)$$
$$(1,0) \cdot (0,2) = (0,0)$$
$$(0,2) \cdot (2,0) = (0,0)$$
$$(2,0) \cdot (0,1) = (0,0)$$

Example 1.4. Consider the ring $\mathbb{Z}_2[x,y]/ < x^2, xy, y^2 >$

The elements of this ring are: 0, 1, x, y, $x+1$, $y+1$, $x+y$, $x+y+1$



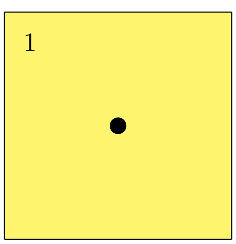
$xy \equiv 0 \bmod xy$

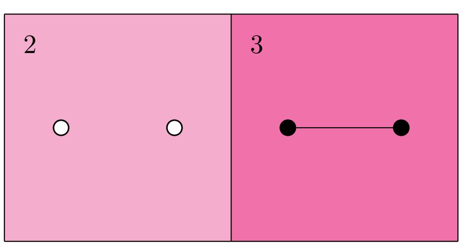
$x(x+y) = x^2 + xy$; $x^2 \equiv 0 \bmod x^2$; $xy \equiv 0 \bmod xy$

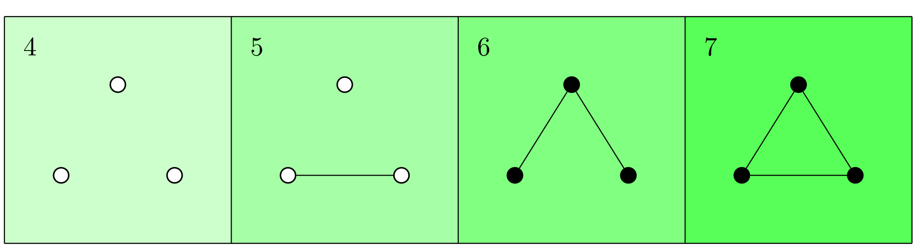
$(x+y)y = xy + y^2$; $xy \equiv 0 \bmod xy$; $y^2 \equiv 0 \bmod y^2$

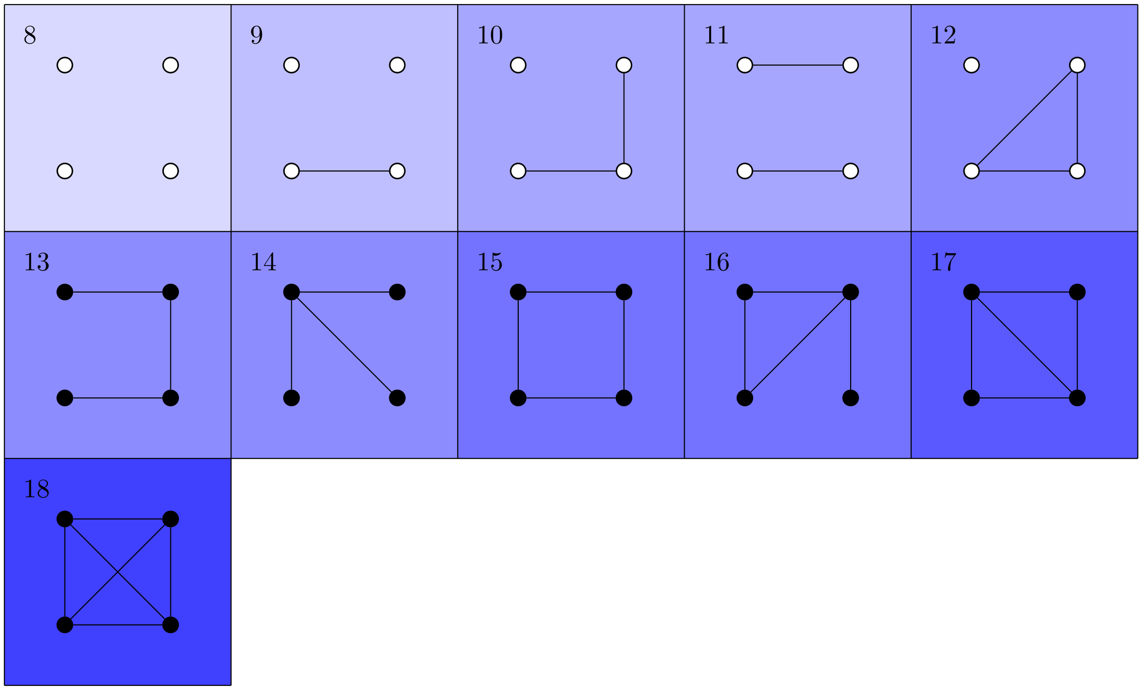
So now that we know how zero divisor graphs work, we can work on the actual problem

1.2. **The Project.** The graphs we were assigned were all the graphs of up to 5 vertices. The table that follows illustrates all of the distint graph from 1 through 5 vertices, leaving us with 52 graphs we have to analyze by hand.





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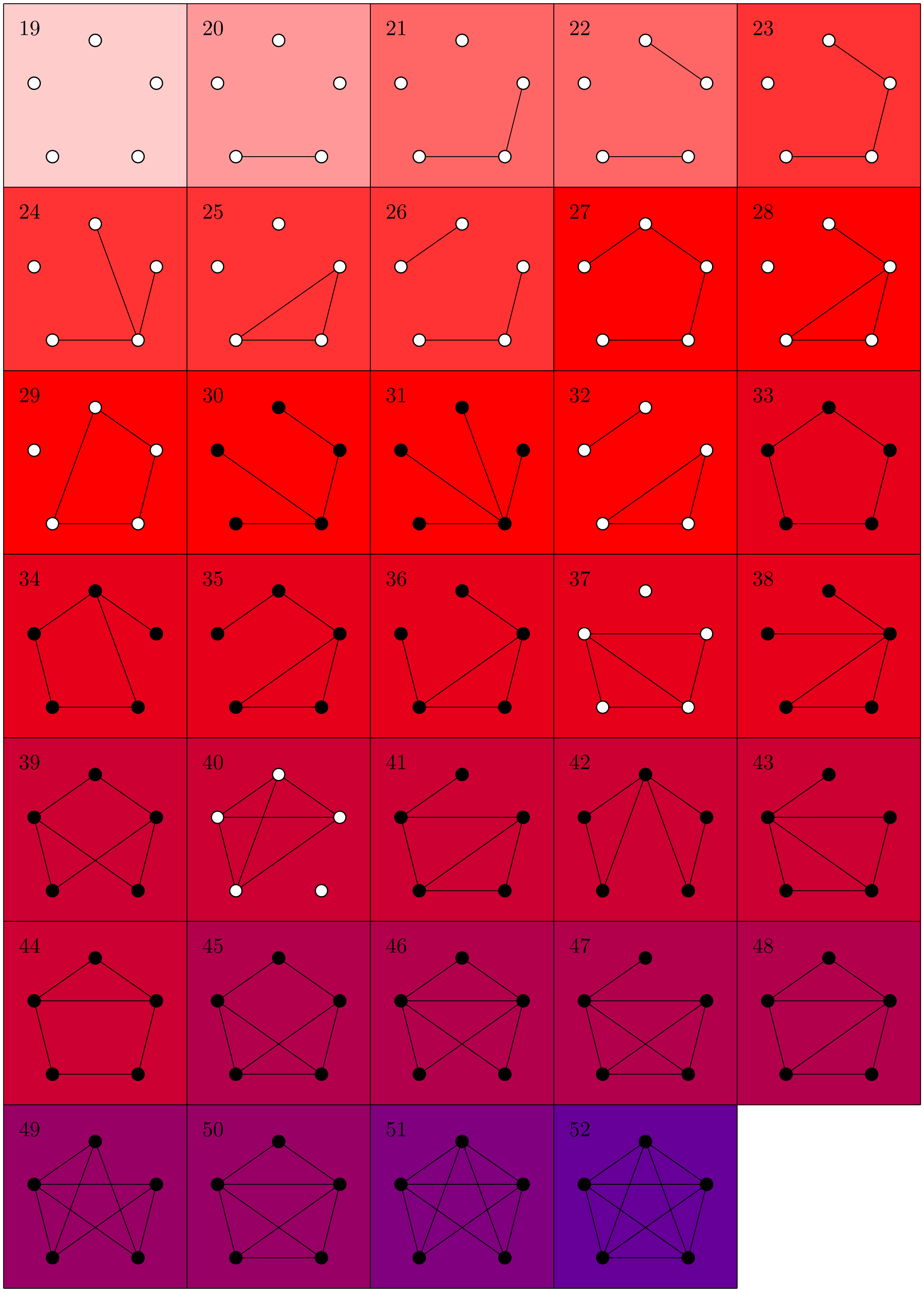
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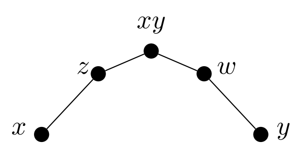
Instead of brute forcing the project and rigorously trying to find rings associated with these graphs, we had to eliminate as much of the graphs as possibly being zero divisor graphs. We resorted to an important theorem proved by Anderson and Livingston which knocked out 22 of the 52 graphs we had to do.

Theorem 1.5. $\Gamma(R)$ *must be a connected graph and have a diameter less than or equal to 3*

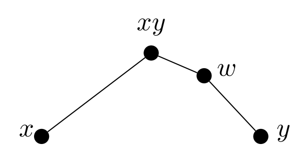
Proof. Assume $\Gamma(R)$ is nonempty and take vertices x and y in $\Gamma(R)$. If x and y are adjacent, we are done.

Case 1. Since x and y are zero divisors, $xz = 0$ and $yw = 0$ for some z,$w \in \Gamma(R)$

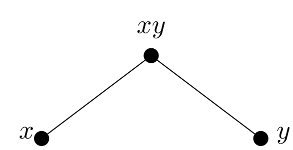
Assume $z \neq x$ and $w \neq y$.



Case 2. $z = x$, $w \neq y$

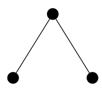
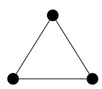
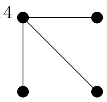


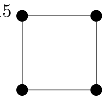
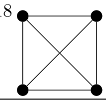
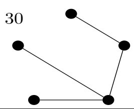
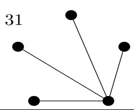
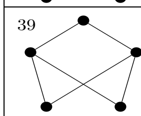
Case 3. $z = x$, $w = y$



As for the diameter, we only have to consider the first case. If $z = w$, then the proof is done. So suppose $z \neq w$. Then if $z \cdot w = 0$, then there is an edge between them and we are done. If $z \cdot w \neq 0$, then we know $z \cdot w$ is a zero divisor and we can find a path $x - z \cdot w - y$ and we are done. $\square$

Now we only have 30 rings to look at instead of 52, lightening the load for the project. Of the 30 remainging graphs, we found examples of commutative rings with unity for 10 of these graphs which are represented in the following table.

1	$\mathbb{Z}_4$, $\frac{\mathbb{Z}_2[x]}{<x^2>}$
3	$\mathbb{Z}_9$, $\mathbb{Z}_2 \times \mathbb{Z}_2$ $\frac{\mathbb{Z}_3[x]}{<x^2>}$
6 	$\mathbb{Z}_6$, $\mathbb{Z}_8$, $\frac{\mathbb{Z}_2[x]}{<x^3>}$
7 	$\frac{\mathbb{Z}_2[x,y]}{<x^2,xy,y^2>}$
14 	$\mathbb{Z}_2 \times \mathbb{F}_4$ $\mathbb{Z}_2 \times \frac{\mathbb{Z}_2[x]}{<x^2>}$

15 	$\mathbb{Z}_3 \times \mathbb{Z}_3$
18 	$\mathbb{Z}_{25}$, $\frac{\mathbb{Z}_5[x]}{<x^2>}$
30 	$\mathbb{Z}_2 \times \mathbb{Z}_4$
31 	$\mathbb{Z}_{10}$
39 	$\mathbb{Z}_3 \times \mathbb{F}_4$

2. Non-Existence Proofs

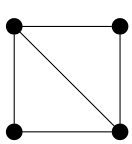
We could not find examples of commutative rings with unity for the remaining 20 graphs, which lead us to believe that these graphs could not be realized as $\Gamma(R)$ for any such R. To say this, however, we had to supply a proof which deduces that this graph could not be ΓR.

These proofs took the form of contradiction proofs so we were able to us the following strategies to disprove these graphs:

- Suppose the given graph is an [AL] zero divisor graph.
- Consider products and sums of the elements
- Look for contradictions that arise with these elements
  - A vertex is equivalent to zero
  - Two vertices are the same
  - A zero divisor appears that is not on the graph

By following these strategies, we found contraditions for the 20 remaining graphs, concluding which graphs of up to 5 vertices can be realized as the zero divisor graph of a commutative ring with unity.

The following are samples of proofs for some of the 20 graphs which could not be realized as the zero divisor graphs of a commutative ring: (it is important to note that each graphs vertices are labelled starting from the bottom left vertex and labelled starting with a going clockwise)

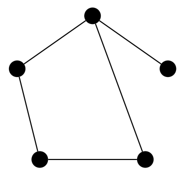


Claim 2.1. *G_{17} cannot be realized as the zero divisor graph of any ring R.*

Proof. Consider the sum $a + d$. Since $a + d$ is annihilated by c, but cannot be annihilated by b, $a + d$ must be equal to b.

Consider the sum $c + d$. Since $c + d$ is annihilated by a, but cannot be annihilated by b, so $c + d$ must be equal to b.

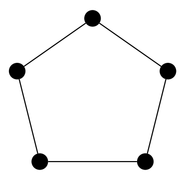
However, this means that $c+d = a+d$, which leads to the contradiction $c = a$. Therefore, G_{17} cannot be realized as the zero divisor graph of any ring R. □



Claim 2.2. *This graph cannot be realized as a zero-divisor graph of a ring*

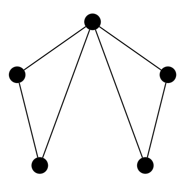
Proof. Consider $b+e$. This element is annihilated by a and d, but not c. Then $b+e$ can only be either b or e. However, if $b+e = b$, then $e = 0$ an contradiction. Likewise, if $b+e = e$, then $b = 0$, another contradiction.

Therfore, this graph cannot be realized as the zero-divisor graph of a ring. □



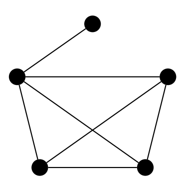
Claim 2.3. *This graph cannot be realized as a zero-divisor graph of a ring.*

Proof. Consider $a \cdot d$, which is annihilated by b, c, and e. Thus $a \cdot d$ is a zero-divisor somewhere on this graph. However, no vertex in this graph has degree greater than 2 so no such vertex exists. Therefore, this graph is not the zero-divsor graph of a ring. □



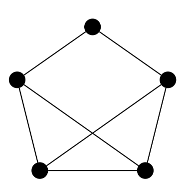
Claim 2.4. *G_{42} cannot be realized as the zero divisor graph of any ring R.*

Proof. Consider the sum $b+e$, which is annihilated by c, but cannot be annihilated by a or d. Since no vertex has these properties, G_{42} cannot be realized as the zero divisor graph of any ring R. □



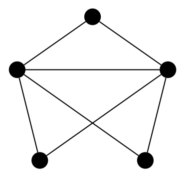
Claim 2.5. *G_{47} cannot be realized as the zero divisor graph of any ring R.*

Proof. Consider the sum $a+c$, which is annihilated by b, but cannot be annihilated by d or e. Since no vertex has these properties, G_{47} cannot be realized as the zero divisor graph of any ring R. □



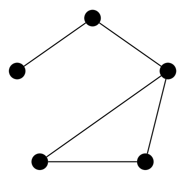
Claim 2.6. *G_{45} cannot be realized as the zero divisor graph of any ring R.*

Proof. Consider the sum $a+b$, which is annihilated by e, but cannot be annihilated by c or d. Since no vertex has these properties, G_{45} cannot be realized as the zero divisor graph of any ring R. $\square$



Claim 2.7. *G_{46} cannot be realized as the zero divisor graph of any ring R.*

Proof. Consider the sum $b+d$, which is annihilated by a, c, and e. Since b and d are the only vertices with these properties, and either possibility would lead to a contradiction, G_{46} cannot be realized as the zero divisor graph of any ring R. $\square$



Claim 2.8. *G_{35} cannot be realized as the zero divisor graph of any ring R.*

Proof. Consider the product cd, which is annihilated by a, b, and e. Since no vertex has these properties, G_{35} cannot be realized as the zero divisor graph of any ring R. $\square$

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