A Survey of Hybrid Control Systems

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Motivation

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- We consider systems where the state can change continuously or discretely and the dynamics themselves can change discretely.

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- We consider systems where the state can change continuously or discretely and the dynamics themselves can change discretely.
- This is a very general "definition" of hybrid systems, so naturally there are several frameworks to consider. We will look at several hybrid control systems.

Standard Control System

A *standard control system* is of the form

$$\dot{x} = f(t, x(t), u(t)), \ x(0) = x_0,$$

where the function $u(t) \in U$ is a control function.

Differential Inclusions

- Let F map $\mathbb{R} \times \mathbb{R}^n$ to the subsets of $\mathbb{R}^n$. Then a differential inclusion is of the form

$$\dot{x}(t) \in F(t, x(t))\text{a.e.}, t \in [a, b]$$

- If we let $F(t, x(t)) = f(t, x(t), U)$, then differential inclusions can subsume the control system formulation
- Thus, differential inclusions cover a wider array of problems.

Continuous Optimal Control

We want to solve $\mathcal{P}_c$:

$$\min_{u(t)\in U} \ell(x(T)) + \int_0^T L(t, x(t), u(t))dt$$

over all $x(t)$ that satisy

$$\dot{x} = f(t, x(t), u(t)),\ x(0) = x_0,\ x(T) \in C_1$$

This is called the Bolza problem. Note we can replace the dynamics with a differential inclusion, which will sometimes be utilized in the following examples. This will lead us to rewrite L so that it depends instead on $(t, x, \dot{x})$.

Pseudo-Hamiltonian

We define the *pseudo-Hamiltonian*, a map $H_p : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}$ given by

$$H_p(t, x, p, u, \lambda) := \langle p, f(t, x, u) \rangle - \lambda L(t, x, u).$$

Pontryagin Maximum Principle

Let (x, u) solve the optimal control problem. Then there is a $\lambda \in \{0, 1\}$, and arc p such that

- The adjoint equation below holds a.e.

$$-\dot{p}(t) \in \partial_x H_p(t, x(t), p(t), u(t), \lambda)$$

- The psuedo Hamiltonian is maximized at $u(t)$ a.e.; i.e.

$$\begin{aligned} & max\{H_p(t, x(t), w(t), \lambda) : w(t) \in U(t)\} \\ = \quad & H_p(t, x(t), p(t), u(t), \lambda) \end{aligned}$$

Pontryagin Maximum Principle

Let (x, u) solve the optimal control problem. Then there is a $\lambda \in \{0, 1\}$, and arc p such that

- $\|p\| + \lambda > 0$.
- There is a $\zeta \in \partial_C \ell(x(b))$ so that the following transversality condition holds:

$$-p(b) - \lambda\zeta \in N^C_{C_1}(x(b)).$$

Hamilton-Jacobi Equations

Using a differential inclusion notation for our system, we define the *Hamiltonian* in the standard way

$$H(x,p) = \sup_{v \in F(t,x)} \{\langle p, v\rangle - L(t,x,v)\}$$

and define the *value function* as

$$V(t,x) = \inf\left(\int_t^T L(s, x(s, u(s))ds + \ell(x(T))\right).$$

Then we can show that

$$V_t = H(x, \nabla_x V).$$

Multiprocesses

Our first hybrid control problems are called multiprocesses. A multiprocess is a k-tuple comprised of $\{\tau_0^i, \tau_1^i, y_i(\cdot), w_i(\cdot)\}$ where the first two entries are the endpoints of a closed interval, and

$$y_i(\cdot) : [\tau_0^i, \tau_1^i] \to \mathbb{R}^{n_i},$$

$$w_i(\cdot) : [\tau_0^i, \tau_1^i] \to \mathbb{R}^{m_i}$$

are absolutely continuous and measurable, respectively. We require

$$\dot{y}_i = f_i(t, y_i(t), w_i(t))$$

Optimal Multiprocesses

Essentially, multiprocesses are ordered set of control systems where we are allowed to choose when we switch between the different systems. We use the cost function

$$\ell(\{\tau_0^i, \tau_1^i, y_i(\tau_0^i), y_i(\tau_1^i)\}) + \sum_i \int_{\tau_0^i}^{\tau_1^i} L_i(t, y_i(t), w_i(t))dt$$

We will require, however, that we be given a set Λ such that

$$\{\tau_0^i, \tau_1^i, y_i(\tau_0^i), y_i(\tau_1^i)\} \subset \Lambda$$

Our problem is then to find a multiprocess that minimizes the cost function with the above endpoint constraing satisfied.

An Example of a Multiprocess

Consider the situation of harvesting a renewable resource. The standard dynamics are given by

$$\dot{x}(t) = F(x(t)) - \sigma x(t)u(t).$$

We restrict u to the interval $[0, E]$. Then a standard profit function which we wish to maximize, including the discount constant δ is

$$\int_0^T e^{-\delta t}[\pi x(t) - c]u(t)dt.$$

An Example of a Multiprocess

We turn to the situation where there are two species x_1, and x_2 which we wish to switch between once in order to optimize profits. We introduce the following cost function

$$\begin{aligned}\phi_0 e^{-\delta\tau} - & \int_0^\tau e^{-\delta t}[\pi_1 x_1(t) - c_1]u(t)dt \\ - & \int_\tau^T e^{-\delta t}[\pi_2 x_2(t) - c_2]u(t)dt\end{aligned}$$

with switching time τ and initial condition will be $x_2(\tau) = z(\tau)$ where

$$\dot{z}(t) = F_2(z(t)), \quad z(0) = x_0^2.$$

A Survey of Hybrid Control Systems
Continuous Optimal Control
Multiprocesses
SGP Systems
Stratified Domains
Teel Hybrid Systems

Multiprocess Maximum Principle

Suppose $\{T_0^i, T_1^i, x_i(\cdot), u_i(\cdot)\}$ is a minimizing multiprocess. Then, under basic assumptions, like those for the PMP, there are are real numbers $\lambda \geq 0$, h_0^i, h_1^i, and absolutely continuous functions $p_i(\cdot) : [T_0^i, T_1^i] \to \mathbb{R}^{n_i}$ such that $\lambda + \sum_i |p_i(T_1^i)| = 1$ and the following hold true

- $-\dot{p}_i(t) \in \partial_{C\ x} H_i(t, x(t), u_i(t), p_i(t), \lambda)$, a.e. $t \in [T_0^i, T_1^i]$,
- $H_i(t, x(t), u_i(t), p_i(t), \lambda) = \max_{w \in U_t^i} H_i(t, x(t), w, p_i(t), \lambda)$a.e. $t \in [T_0^i, T_1^i]$,

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- $h_0^i \in \text{co ess}_{t \to T_0^i}[\sup_{w \in U_t^i} H_i(t, x(T_0^i), w, p_i(T_0^i), \lambda)]$,
- $h_1^i \in \text{co ess}_{t \to T_1^i}[\sup_{w \in U_t^i} H_i(t, x(T_1^i), w, p_i(T_1^i), \lambda)]$,
- $\{-h_0^i, h_1^i, p(T_0^i), -p(T_1^i)\} \in N_\Lambda^C + \lambda \partial^C f$.

Our Example Revisited

The tranversality conditions state that

$$h_0^2 = h_1^1 + p_2(\tau)\dot{z}(\tau) + \delta\phi_0 e^{-\delta t}$$

Then, using knowledge from the PMP for the one system case, we know that u will be maximal on both sides of τ so

$$h_0^2 = p_2(\tau)[F_2(z(\tau)) - \sigma_2 z(\tau)E] + e^{-\delta\tau}[\pi_2 z(\tau) - c_2]E$$

and

$$h_1^1 = e^{-\delta\tau}[\pi_1 x_1(\tau) - c_1]E.$$

Our Example Revisited

We then get the following implicit statement on the switching time using only necessary conditions

$$\begin{aligned}[\pi_2 x_2(\tau) - c_2]E &= [\pi_1 x_1(\tau) - c_1]E + \delta\phi_0 \\ &+ e^{\delta\tau} p_2(\tau)\sigma_2 x_2(\tau)E.\end{aligned}$$

SGP Systems

We now turn to a new hybrid system which can be seen as generalizing, in some ways, autonomous multiprocesses by removing the ordering of the switches. We are given the following data

- A finite set Q,
- A family of smooth manifolds $M = \{M_q\}_{q \in Q}$ and sets $U' = \{U'_q\}_{q \in Q}$,
- Functions $f_q : M_q \times U_q \to TM_q$ with $f_q(x, u) \in T_x M_q$ for each $(x, u) \in MQ \times U'_q$,

SGP Systems

- $U = \{U_q\}_{q \in Q}$, a family of sets of maps from $\mathbb{R}$ into U'_q,
- A family of intervals $J = \{J_q\}_{q \in Q}$ where $J_q \subset \mathbb{R}^+$.
- A subset S of

$$\{(q, x, q', x', u(\cdot), \tau) : q, q' \in Q,\\ x \in M_q, x' \in M_{q'}, u(\cdot) \in \mathcal{U}_{q'}, \tau \in J_{q'}\}.$$

SGP Systems

We define a solution as a triple $X(t) = (q(t), x(t), \tau(t))$ where there is a $\{t_i\}$ partition of $[0, T]$ such that

- If $x_i(\cdot) = x|_{(t_i, t_{i+1})}$, then $\dot{x}_i(t) = f_{q_i}(x(t), u(t))$
- $(x_i(t_i), x_{i+1}(t_i)) \in S_{q_i, q_{i+1}}$ where

$$\begin{aligned} S_{q_i,q_{i+1}} \quad := \quad & \{(x, x') \in M_{q_i} \times M_{q_{i+1}} : \\ & (q, x, q', x', u(\cdot), \tau) \in S \\ & \text{ for some } u(\cdot) \in U_{q_{i+1}}, \tau \in J_{q_{i+1}}\} \end{aligned}$$

SGP Systems

Also, we need

- $u_{i+1} \in U_{q_i, x_i(t_i), q_{i+1}, x_{i+1}(t_i)}$ where

$$U_{q_i, x_i(t_i), q_{i+1}, x_{i+1}(t_i)} := \{u(\cdot) \in U_{q_{i+1}} : (q_i, x_i, q_{i+1}, x_{i+1}, u(\cdot), \tau) \in S \text{ for some } \tau \in J_{q_{i+1}}\}$$

SGP Cost Functions

The cost function associated with this problem is of the following form

$$\begin{aligned} C(X) &= \sum_{j=1}^{\nu} \int_{t_{j-1}}^{t_j} L_{q_j}(x_j(t), u_j(t))dt \\ &+ \sum_{j=1}^{\nu-1} \Phi_{q_j, q_j+1}(x_j(t_j), x_{j+1}(t_j)) \\ &+ +\phi_{q_1, q_\nu}(x_1(t_0), x_\nu(t_\nu)). \end{aligned}$$

Hybrid Maximum Principle

Let X be a solution to the above hybrid problem. Then there is an adjoint pair (p, λ) with $p = \{p_1, p_2, \ldots, p_\nu\}, \; \lambda \in \mathbb{R}^+$ such that

- $-\dot{p}(t) = \partial_x H_i(t, x(t), p(t), u(t), \lambda)$
- $-\dot{p}(t) = \partial_x H_i(t, x(t), p(t), u(t), \lambda)$
- The Hamiltonian is maximized for this adjoint pair

Hybrid Maximum Principle

Let X be a solution to the above hybrid problem. Then there is an adjoint pair (p, λ) with $p = \{p_1, p_2, \ldots, p_\nu\},\ \lambda \in \mathbb{R}^+$ such that

- The switching condition below holds

$$(-p_i(t_i), p_{i+1}(t_i)) - \lambda \nabla \Phi_{q_i, q_{i+1}}(x_i(t_i), x_{i+1}(t_i)) \in T^C_{S_{q_i, q_{i+1}}}$$

Hybrid Maximum Principle

Let X be a solution to the above hybrid problem. Then there is an adjoint pair (p, λ) with $p = \{p_1, p_2, \ldots, p_\nu\}, \ \lambda \in \mathbb{R}^+$ such that Also

- if $t_i - t_{i-1} \in \text{Int}(J_{q_i})$, then $\sup H_i = \sup H_\nu = 0$
- if $t_i - t_{i-1}$ is the left endpoint of a nontrivial J_{q_i}, then $\sup H_i \leq 0$
- if $t_i - t_{i-1}$ is the right endpoint of a nontrivial J_{q_i}, then $\sup H_i \geq 0$

Stratifications of $\mathbb{R}^n$

Assume that we have a finite set of disjoint embedded submanifolds $M_j \subset \mathbb{R}^n$ whose union is $\mathbb{R}^n$. Furthermore if $M_j \cap \overline{M_k} \neq \emptyset$, then $M_j \subset \overline{M_k}$.

Infinite horizon problem

Suppose our cost function is of the form

$$\int_0^{\inf} e^{-\beta t} L(x(t), u(t)) dt$$

with dynamics

$$\dot{x}(t) = f(x(t), u(t) \quad x(0) = x_0.$$

Assumptions

We assume that on each submanifold M_i our controls are a compact set U_i, we have a continuous $f_i : M_i \times U_i \to \mathbb{R}^n$ and a cost L_i so that f_i is Lipschitz in the state variable, L_i is nonnegative and continuous. Finally, $f(x, u) = f_i(x, a)$ and $L(x, u) = L - i(x, u)$ when $x \in M_i$.

Hamilton-Jacobi Equation

If we define a multifunction in an analagous manner, and take its convex, semicontinuous regularization,it can be shown that the value function solves the Hamilton-Jacobi equation.

Hybrid Time Domains

An alternative model of hybrid dynamics uses differential inclusions. First, we define a *compact hybrid time domain* as

$$\bigcup_{j=0}^{J-1}([t_j, t_{j+1}] \times \{j\}.$$

And a *hybrid time domain* E is such that for any T, J then $E \cap [0, T] \times \{0, 1, \ldots, J\}$ is a compact hybrid time domain.

Dynamics of Teel System

We give our hybrid dynamics as

$$\begin{aligned} \dot{x} &\in F(x) \text{ when } x \in C \\ x^{+} &\in G(x) \text{ when } x \in D \end{aligned}$$

We refer to F and C as the *flow map* and *flow set*. Similarly, G and D are the *jump map* and *jump set*.

Hybrid Arcs

A *hybrid arc* is a hybrid time domain and a function x such that

$$\dot{x}((t,j) \in F(x(t,j)) \;\; \text{if } x(t,j) \in C$$

on the interval (t_j, t_{j+1}) and

$$x(t, j+1) \in G(x(t,j))$$

if $x(t,j) \in D$ and $(t,j), (t,j+1) \in \text{dom } x$.

Example: The Bouncing Ball

A bouncing ball can be modeled easily with this framework. If x is the height of the ball above the floor, let

$$y = \begin{bmatrix} x \\ \dot{x} \end{bmatrix},$$

$$f(y) = \begin{bmatrix} y_2 \\ -g \end{bmatrix}$$

and

$$C = \{y_1 > 0 \text{ or } y_1 = 0 \text{ and } y_2 > 0\}.$$

Example: The Bouncing Ball

Then we model the jump condition by setting

$$D = \{y_1 = 0 \text{ and } y_2 \leq 0\}$$

and $G(y) \begin{bmatrix} 0 \\ -\mu y_2 \end{bmatrix}$. where $\mu \in (0,1)$ is a dissipation factor.