Introduction to convex sets

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References

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- Borwein, J.M. and Lewis, A.S. *Convex analysis and nonlinear optimization*. Springer Verlag, N.Y., 2000.
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Notation

- We work in a n-dimensional real Euclidean space E.
- Sets will be indicated with capital letters.
- Points and vectors will be lower case.
- For scalars we use greek characters.

Definition

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  (c) Hyperplanes

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- General form

$$M = a + L,$$
$$a \in M,\ L \text{ subspace}$$
$$\dim(M) = \dim(L)$$

Affine hull

For $S \subset E$,

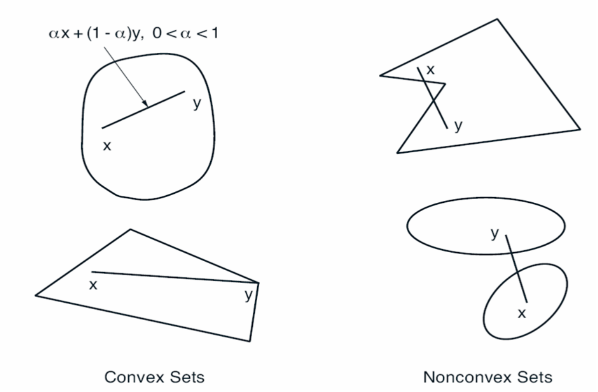
$$\begin{aligned} \mathrm{aff}\,(S) &= \text{smallest affine set containing } S \\ &= \left\{ \sum_{i=1}^{m} \lambda_i x_i \mid x_i \in S, \sum_{i=1}^{m} \lambda_i = 1, m \in \mathbb{N} \right\} \end{aligned}$$

We define

$$\dim\,(S) = \dim\,\mathrm{aff}\,(S)$$

Definition

$C \subset E$ is convex if for any two points x and y in C, the *line segment* passing through x and y is contained in C.



Properties

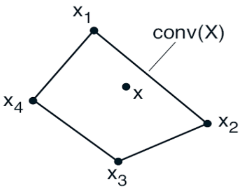
Set operations that preserve convexity

- Arbitrary intersections.
- Scalar multiplication.
- Vector sum.
- Image and inverse image under linear and affine transformations.

Convex hull

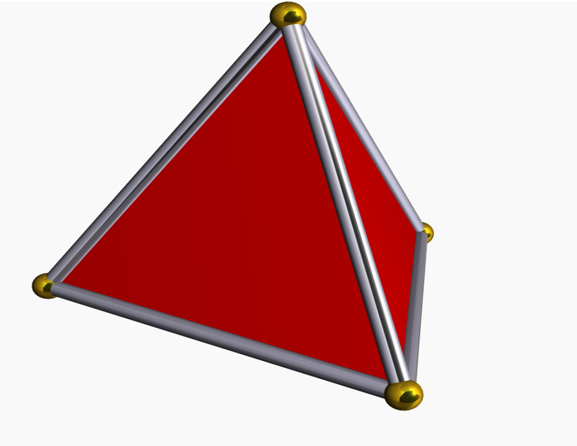
For $X \subset E$,

$$\begin{aligned} \operatorname{conv}(X) &= \text{smallest convex set containing } X \\ &= \left\{ \sum_{i=1}^{m} \lambda_i x_i \mid x_i \in X, \lambda_i \geq 0, \sum_{i=1}^{m} \lambda_i = 1, m \in \mathbb{N} \right\} \end{aligned}$$



Simplices

An m-dimensional simplex is the convex hull of $m+1$ affinely independent vectors in E.



Simplices and convex sets

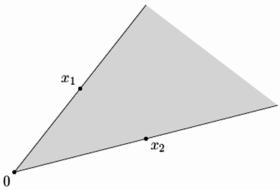
Theorem

The dimension of a convex set C is the largest dimension of the various simplices contained in C.

Definition

$K \subset E$ is a cone if it is nonempty and closed under positive scalar multiplication.

$K \subset E$ is a convex cone if it is a cone and it is convex $\Longleftrightarrow$ K is closed under addition and positive scalar multiplication



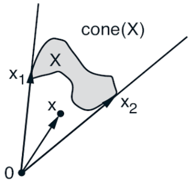
Examples of convex cones

(a) Nonnegative orthant $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n \mid \text{each } x_i \geq 0\}$.
(b) Positive orthant $\mathbb{R}^n_{++}$.
(c) Open and closed half-spaces determined by hyperplanes passing through the origin.
(d) Cone of vectors with nonincreasing components $\mathbb{R}^n_{\geq} = \{x \in \mathbb{R}^n \mid x_1 \geq x_2 \geq \cdots \geq x_n\}$.

Cone generated by a set

For a set $X \subset E$,

$$\begin{aligned} \text{cone}\,(X) &= \{\text{smallest convex cone containing}\, X\} \cup \{0\} \\ &= \left\{ \sum_{i=1}^{m} \lambda_i x_i \mid x_i \in X, \lambda_i \geq 0\,, m \in \mathbb{N} \right\} \end{aligned}$$



Proposition

For a set $S \subset E$,

$$aff(S) = aff(conv\, S)$$

Proposition

Let K be a convex cone containing the origin (in particular, the condition is satisfied if $K = cone(X)$, for some X). Then

$$\begin{aligned} aff(K) &= K - K \\ &= \{x - y \mid x, y \in K\} \end{aligned}$$

is the smallest subspace containing K and $K \cap (-K)$ is the smallest subspace contained in K.

Proposition

Every convex set $C \subset E$ can be regarded as a cross-section of a convex cone $K \subset E \times \mathbb{R}$

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Proof.

Define

$$\begin{aligned} K &= \operatorname{cone}\{(x,1) \mid x \in C\} \\ &= \{(\lambda x, \lambda) \mid \lambda > 0, x \in C\} \cup \{0\} \end{aligned}$$

Then C can be identified with the intersection between K and the hyperplane $\{(y, \lambda) \mid \lambda = 1\}$ $\square$

Theorem (Part 1)

Let X be a nonempty subset of E. Then

(a) *Every nonzero x in cone(X) can be represented as a positive linear combination of vectors $x_1, \ldots, x_m$ from X that are linearly independent.*

Sketch of proof.

Let m be the smallest integer so that x is a positive linear combination of elements $x_1, \ldots, x_m$ from X and prove by contradiction that $x_1, \ldots, x_m$ are linearly independent. □

Theorem (Part 2)

Let X be a nonempty subset of E. Then

(b) *Every $x \in conv(X) \setminus X$ can be represented as a convex combination of vectors $x_1, \ldots, x_m$ from X that are affinely independent.*

Therefore, a vector in cone(X) (respect. conv(X)) may be represented by no more than n (respect. $n+1$) vectors in X.

Sketch of proof.

Apply Part 1 to $Y = \{(x, 1) \mid x \in X\}$. $\square$

Corollary

Let X be a nonempty compact subset of E. Then conv(X) is also a compact subset of E.

However, cone(X) might fail to be closed even if X is compact.

If the set X is just closed, conv(X) is not necessarily closed.