Introduction to convex sets II: Convex Functions

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References

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Notation

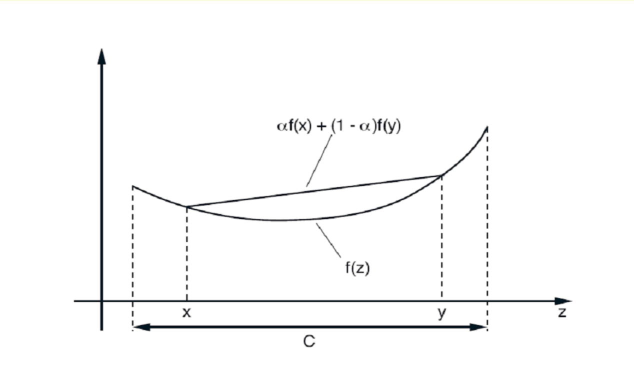
- We work in a n-dimensional real Euclidean space E.
- Sets will be indicated with capital letters.
- Points and vectors will be lower case.
- For scalars we use greek characters.

Convex functions

Let $C \subset E$ be a convex set. A function $f : C \to \mathbb{R}$ is convex if

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y)$$

for all $x, y \in C$ and $0 < \alpha < 1$.

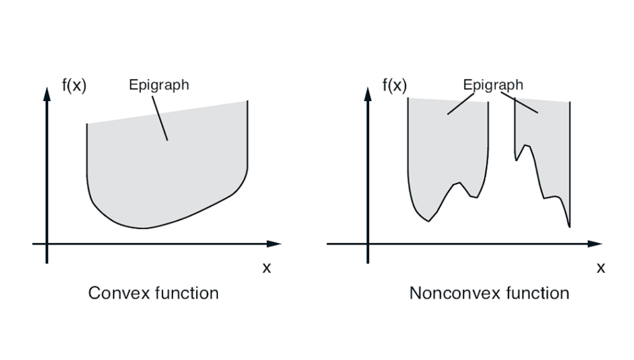


Geometric interpretation

Let $C \subset E$ be a convex set. A function $f : C \to \mathbb{R}$ is convex if and only if the set

$$\operatorname{epi} f = \{(x, r) \mid r \geq f(x)\}$$

is convex as a subset of $E \times \mathbb{R}$.



Extended definition of convex function

Definition

A function $\tilde{f} : E \to [-\infty, +\infty]$ is convex if its epigraph is a convex set in $E \times \mathbb{R}$.

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- Simpifies notation.
- The supremum of a set of functions might take infinite values, even if all the functions in the set are finite.
- Allows penalization and exclusion in optimization problems.

Properness

Definition

A extended-valued function $\tilde{f}$ is called *proper* provided

- $\tilde{f}$ is not identically $+\infty$
- $\tilde{f}(x) > -\infty$, for all x.

Proper functions help us avoid undefined expressions such as $+\infty - \infty$.

Extension of a finite-valued convex function on C as a extended-valued convex function on E

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \in C, \\ +\infty & \text{otherwise.} \end{cases}$$

Restriction of a extended-valued proper convex function on E to a finite-valued convex function

Take

$$C = \operatorname{dom} \tilde{f} = \left\{ x \,|\, \tilde{f}(x) < +\infty \right\}$$

and define

$$f : C \to \mathbb{R}, f = \tilde{f}_{|C}$$

Jensen's inequality

Let $f : E \to (-\infty, +\infty]$ be a function. Then f is convex if and only if

$$f(\sum_{i=1}^{m} \lambda_i x_i) \leq \sum_{i=1}^{m} \lambda_i f(x_i)$$

whenever $\lambda_i \geq 0$, for all i, $\sum_{i=1}^{m} \lambda_i = 1$.

Convex functions on the real line

For g a real-valued function on an interval I.

Proposition

g is convex on I if and only if, for all $x_0 \in I$, the slope-function

$$x \mapsto \frac{f(x) - f(x_0)}{x - x_0}$$

is increasing in $I \setminus \{x_0\}$.

Proposition

If g is convex on I, then g is continuous on the interior of I.

Convex functions on the real line

Proposition

If g is convex on I, then g admits finite left and right derivatives at each x_0 in the interior of I.

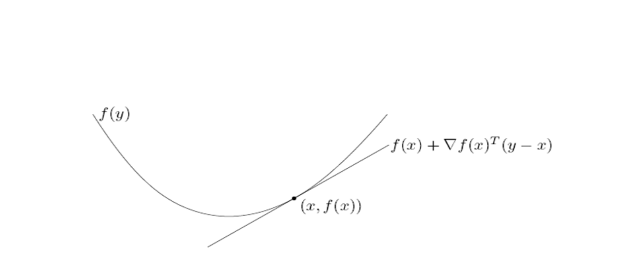
First order condition

Theorem

Let $f : E \to [-\infty, +\infty]$ be a differentiable function. Then f is convex if and only if $\operatorname{dom} f$ is a convex set and

$$f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle$$

for every $x, y \in \operatorname{dom} f$.



Necessary and sufficient condition for optimality

Corollary

Let $f : E \to [-\infty, +\infty]$ be a differentiable convex function. Then $x \in dom\, f$ is a global minimizer if and only if $\nabla f(x) = 0$

Corollary

Let $f : E \to [-\infty, +\infty]$ be a differentiable convex function. Then the mapping ∇f is monotone, i.e.,

$$\langle \nabla f(y) - \nabla f(x), y - x \rangle \geq 0, x, y \in dom\, f$$

Sketch of proof

(1) Show the result for $g : \mathbb{R} \to [-\infty, +\infty]$.

(2) Use the fact that $f : E \to [-\infty, +\infty]$ is convex if and only if the real function g defined by

$$g(t) = f(ty + (1-t)x), \ ty + (1-t)x \in \operatorname{dom} f$$

is convex.

Second order condition

Theorem

Let f be a twice continuously differentiable real-valued function on an open interval (α, β). Then f is convex if and only if its second derivative is nonnegative throughout (α, β).

Theorem

Let $f : E \to [-\infty, +\infty]$ be a twice continuously differentiable function. Then f is convex if and only if $\operatorname{dom} f$ is a convex set and the Hessian matrix $\nabla^2 f(x) \succeq 0$ for all $x \in \operatorname{dom} f$.

Examples of convex functions in the real line

- $g(x) = \exp(\alpha x), x \in \mathbb{R}$
- $g(x) = x^p, 1 \leq p < \infty, x \leq 0$
- $g(x) = |x|^p, 1 \leq p < \infty$
- $g(x) = -x^p, 0 \leq p < 1, x \leq 0$
- $g(x) = x^p, -\infty < p < 0, x > 0$
- $g(x) = \left(\alpha^2 - x^2\right)^{-1/2}, \alpha > 0, |x| < \alpha$
- $g(x) = -\log(x), x > 0$
- Negative entropy $g(x) = x\log(x), x > 0$

Examples of convex functions in $\mathbb{R}^n$

- Any norm
- $f(x) = \max\{x_1, x_2, \ldots, x_n\}$
- Log-sum-exp $f(x) = \log(\exp(x_1) + \exp(x_2) + \cdots + \exp(x_n))$
- Geometric mean $f(x) = \left(\prod_{i=1}^n x_i\right)^{1/n}$
- Indicator function of a convex set C , $\delta(\cdot \mid C)$

$$\delta(x \mid C) = \begin{cases} 0 & \text{if } x \in C, \\ +\infty & \text{otherwise.} \end{cases}$$

We have

$$\inf_{x \in C} f(x) = \inf_{x \in E} \left(f(x) + \delta(x \mid C)\right)$$

Operations that preserve convexity

Suppose $f, f_1, \ldots, f_m$ are convex functions on E

- $h(x) = \lambda_1 f_1 + \cdots + \lambda_m f_m,\ \lambda_i$ are positive scalars.
- $h(x) = \sup\{f_1(x), \ldots, f_n(x)\}$.
- $h(x) = f(Ax), A$ linear transformation.
- Inf-convolution
  $h(x) = (f_1 \star f_2)(x) = \inf_{y \in E}\{f_1(x-y) + f_2(y)\},\ f_1, f_2$ proper

Applications to inequalities

Convexity of $-\log(x)$ ensures that, for $0 < \theta < 1, \; a, b \geq 0$

$$a^\theta b^{1-\theta} \leq \theta a + (1-\theta) b$$

A particular selection for a and b helps proving Hölder's inequality: for $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$,

$$\sum_{i=1}^{n} |x_i y_i| \leq \left(\sum_{i=1}^{n} x_i^p \right)^{1/p} \left(\sum_{i=1}^{n} y_i^q \right)^{1/q}, \text{ where } 1/p + 1/q = 1$$

More results: Level sets

Proposition

For a convex function f on E, the level sets

$$\{x \mid f(x) < \alpha\} \text{ and } \{x \mid f(x) \leq \alpha\}$$

are convex for every α.

Note: reverse does not hold!

Corollary

For an arbitrary family $\{f_i\}$ of convex functions on E and real numbers $\alpha_i, i \in I$, the set

$$\{x \mid f_i(x) \leq \alpha_i, i \in I\}$$

is convex.

Existence of global minimizers

Proposition

Let $D \subset E$ be nonempty and closed, and that all the level sets of the continuous function $f : D \to \mathbb{R}$ are bounded. Then f has a global minimizer.

Proposition

For a convex $C \subset E$, a convex function $f : C \to \mathbb{R}$ has bounded level sets if and only if it satisfies the growth condition

$$\liminf_{||x|| \to \infty} \frac{f(x)}{||x||} > 0$$