Introduction to Differential Inclusions

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References

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Multifunctions

A *multifunction* $F : \mathbb{R}^m \to \mathbb{R}^n$ is a map from $\mathbb{R}^m$ to the subsets of $\mathbb{R}^n$, that is for every $x \in \mathbb{R}^m$, we associate a (potentially empty) set $F(x)$.
Its graph, denoted $Gr(F)$ is defined by

$$Gr(F) = \{(x, y) | y \in F(x)\}.$$

Measurability

A multifunction $F : S \to \mathbb{R}^n$ is *measurable* if for every open (closed) $C \subseteq \mathbb{R}^n$,

$$\{x \in S : F(x) \cap C \neq \emptyset\}$$

is Lebesgue measurable.

Continuity

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- A multifunction F is called *lower semi-continuous* at x_0 if for any $y_0 \in F(x_0)$ and any neighborhood M of y_0 there is a negihborhood Ω of x_0 so that

$$F(x) \cap M \neq \emptyset, \quad \forall x \in \Omega.$$

A multifunction is *continuous* at x_0 if it is both upper and lower semi-continuous at x_0.

Lipschitz Continuity

A multifunction F is said to be *Lipschitz continuous* if there is a $k \geq 0$ so that for any $x_1, x_2 \in \mathbb{R}^m$ we have

$$F(x_1) \subset F(x_2) + k|x_1 - x_2|B.$$

The selection problem

Given a multifunction $F : \mathbb{R}^m \to \mathbb{R}^n$, a single-valued map $f : \mathbb{R}^m \to \mathbb{R}^n$ is a *selection* if

$$f(x) \in F(x), \quad \forall x \in \mathbb{R}^m.$$

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For what multifunctions are we assured of the existence of a selection?

Michael's Selection Theorem and a Measurable Selection Theorem

Theorem

Let F be a closed, convex, and lower semi-continuous multifunction. Then there is a continuous selection from F.

Theorem

Let F be measurable, closed, and nonempty on S. Then there is a measurable selection from F.

Approximate Selections

Theorem

Let F be a convex, upper semi-continuous multifunction. Then for $\epsilon > 0$ there is a locally Lipschitz continuous function f_ϵ whose range is in the convex hull of the range of F and

$$Gr(f_\epsilon) \subset Gr(F) + \epsilon B.$$

Differential Inclusions

We now turn our attention to the problem of solving *Differential Inclusions*:

$$\dot{x}(t) \in F(x(t)), \quad t \in [0, T], \tag{1}$$

with $x(0) = x_0$. We will assume that F is closed, convex, and Lipschitz continuous with constant $k > 0$. We shall see presently that these assumptions are not too restrictive, especially when we concern ourselves with problems from control theory.

An Example from Control Theory

Consider the control system

$$\dot{x} = f(x, u)$$

where $u \in U \subset \mathbb{R}^k$. We assume that f is Lipschitz in x, and we allow any measurable functions $u : [0, t] \to \mathbb{R}^k$ so that $u(t) \in U$ a.e. and that for all x, $f(x, u(t)) \in L_1([0, T])$.

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$$F(x) = f(x, U) = \bigcup_{u \in U} f(x, u).$$

We do the same for nonautonomous systems using the right-hand side $F(t, x(t))$.

Filippov's Theorem

Theorem

Let $f : \mathbb{R}^m \times \mathbb{R}^k \to \mathbb{R}^n$ *be continuous, and let* $v : \mathbb{R}^m \to \mathbb{R}^n$ *be measurable. Assume* U *is compact so that* $v(x) \in f(x, U)$ *a.e. Then there is a measurable* $u : \mathbb{R}^m \to U$ *satisfying* $v(x) = f(x, u(x))$.

Filippov's Theorem

Sketch of Proof: We know that

$$U(x) = \{u \in U |\ v(x) = f(x, u)\}$$

has compact values. Let $u(x) = (u^1(x), \ldots, u^k(x)) \in U(x)$ with $u^1(x)$ the smallest possible. We show that if $u^i(x)$ is measurable on a compact set A for $i < p$ then $u^p(x)$ is as well. Then we use the Lusin theorem to find a set A_ϵ where the $u^i(x)$'s and $v(x)$ are continuous and $m(A \setminus A_\epsilon) \leq \epsilon$. We then show that the sublevel sets of $u^p(x)$ restricted to A_ϵ are closed. So $u^p(x)$ is measurable on A_ϵ and we then repeat Lusin on this set to get an $A_{2\epsilon}$. Since ϵ is arbitrary we use Lusin again and see it is measurable on A.

Filippov's Theorem

Theorem

Let $f : \mathbb{R}^m \times \mathbb{R}^k \to \mathbb{R}^n$ be continuous, and let $v : \mathbb{R}^m \to \mathbb{R}^n$ be measurable. Assume U is compact so that $v(x) \inf(x, U)$ a.e. Then there is a measurable $u : \mathbb{R}^m \to U$ satisfying $v(x) = f(x, u(x))$.

Obviously, trajectories of the control system are solutions to the differential inclusion. This result means the converse holds and so the system and the inclusion are equivalent.

Continuity and Relaxation of Differential Inclusions

Proposition

Let U be compact and $f : \mathbb{R}^m \times U \to \mathbb{R}^n$ be continuous (Lipschitz) in x. Then $F : x \to f(x, U)$ is continuous (Lipschitz).

Continuity and Relaxation of Differential Inclusions

Introduction to Differential Inclusions

Definitions

Selections

Differential Inclusions

Proof.

We know that F has closed graph and is locally bounded. Let N be a neighborhood of $F(x)$ with sequences $\{x_i\}$, $\{y_i\}$ so that $x_i \rightarrow x_0$, $y_i \in F(x_i)$, and $y_i \notin N$. Since $cl\{F(x) | \; |x - x_0| < \delta\}$ is compact, WLOG assume $y_i \rightarrow y_0$. But the graph is closed, so $y_0 \in F(x_0)$. So F is upper semi-continuous. The lower semi-continuity follows from the continuity of $x \rightarrow f(x, u), \;\; \forall u \in U$. The Lipschitz statements are similarly straightforward. $\square$

Continuity and Relaxation of Differential Inclusions

A *trajectory* is an absolutely continuous function $x : [a, b] \to \mathbb{R}^n$ such that

$$\dot{x}(t) \in F(t, x(t)) \text{ a.e.}$$

We say F is *integrably bounded* if there is an integrable function $\phi(\cdot)$ such that $|v| \leq \phi(t)$ for all v in $F(t, x)$.

Proposition

A relaxed trajectory y is a trajectory for coF. That is, $\dot{y}(t) \in coF(t, y(t))$. If F is integrably bounded then any relaxed trajectory y is within δ of a trajectory for F in the sup norm.

Continuity and Relaxation of Differential Inclusions

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Proposition

A relaxed trajectory is a trajectory for coF. If F is integrably bounded then any relaxed trajectory y is within δ of a trajectory for F in the sup norm.

Existence Theorem

Theorem

Assume that $F(x)$ is closed, convex, and Lipschitzian. For any $x_0 \in \mathbb{R}^n$ there exist solutions to (1) with $x(0) = x_0$.