Möbius transformations and Parallel Transports in the Upper-Half Plane

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1 Course Description

In the Euclidean plane $\mathbb{R}^2$, the notion of parallel vectors is a natural concept. We study the parallel transport in hyperbolic geometry, in particular for the upper-half plane $\mathbb{R}^2_+$. Along the way, we study covariant derivatives, geodesics, isometries, and Gaussian curvature. The isometries are generated by reflections about lines or semicircles perpendicular to the x-axis, and are called Möbius transformations.

Also, we study $\mathbb{R}^3_+$ and its Möbius transformations. The latter are linear fractional transformations that depend on 2-by-2 Vahlen matrices. The matrix entries satisfy special properties, and lie in a Clifford algebra of $\mathbb{R}^3$. I intend for this course to be computational in nature.

For the projects, Mathematica should be helpful in the calculations and graphic illustrations.

2 Textbook

1. Lecture notes provided by the instructor.

2. Recommended books, but not required: *Riemannian Geometry* by Do Carmo and Flaherty, and *The Geometry of Discrete Groups* by Beardon.

3 Undergraduate Projects

1. *Undergraduate Project #1*

   Describe the parallel transport of tangent vectors along hyperbolic triangles whose sides are geodesics in the upper-half plane $\mathbb{R}^2_+$. Analyze the angles between the initial vector and the transported vector.

   Also, describe the parallel transport along Euclidean lines $y = mx + 1$.

2. *Undergraduate Project #2*

   Let $\triangle ABC$ be a hyperbolic triangle in $\mathbb{R}^2_+$. Describe the isometries that map $\triangle ABC$ to $\triangle A'B'C'$ where $A'(0,1)$, $C'(0,c)$ for some $c > 0$, and B' is some point in $\mathbb{R}^2_+$.

   Extend the above result to hyperbolic triangles in $\mathbb{R}^3_+$.

The Umbral Calculus

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1 Course Description

If $(a_n : n \geq 0)$ is a sequence of real numbers, its *exponential generating function* is

$$\sum_{n=0}^{\infty} a_n \frac{x^n}{n!}.$$

One of the simplest (and most important) sequences in mathematics is the sequence of powers a^n, where a is a real number, whose exponential generating function e^{ax} explains the "exponential" in the terminology.

The Umbral Calculus arose in the 19-th century with the observation (by John Blissard in 1861) that many derivations of formulas involving power series could be greatly simplified when the sequence a_n under study was formally replaced with the powers A^n of an unspecified (and somewhat mysterious) variable A, then algebraic manipulations were performed on resulting exponential generating functions involving A, and finally the powers A^n were again replaced with the sequence a_n. The "shady" nature of the variable A was reflected in its name, *umbra*, that is Latin for "shadow".

For over a century, the Umbral Calculus was considered to be an unjustified set of empirical rules that could be employed to "guess" formulas that must then be rigorously proved using traditional methods. In the 1970's, the subject was given a rigourous foundation by Gian-Carlo Rota and others, and today nothing mysterious remains about the way the umbral approach greatly simplifies the study of many formulas in Combinatorics and other subjects.

As a simple illustration of the method, suppose that a sequence a_n is given, and another sequence b_n is defined by

$$b_n = \sum_{k=0}^{n} \binom{n}{k} a_k. \qquad (1)$$

We wish to invert the above relation and find a_n in terms of b_n. Substituting the umbrae A^n and B^n for a_n and b_n in (1) gives

$$B^n = \sum_{k=0}^{n} \binom{n}{k} A^k = (1 + A)^n.$$

Hence $B = 1 + A$, or $B - 1 = A$, and so

$$A^n = (B-1)^n = \sum_{k=0}^{n} \binom{n}{k} B^k (-1)^{n-k}.$$

Substituting back a_n and b_n for A^n and B^n gives the desired inverse relation:

$$a_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} b_k. \tag{2}$$

It is not hard to check using traditional (but longer) methods that (2) is correct.

In this course, after discussing some necessary background, we will introduce the foundations of the Umbral Calculus, and illustrate the power of the method by deriving many formulas of interest in Combinatorics. Emphasis will be given to concrete examples and computations with *Mathematica*. Much of the material presented here is taken from the excellent paper by Ira Gessel "Applications of the Classical Umbral Calculus", Algebra univers., **49** (2003) 397–434.

2 Textbook and technology

1. Notes and handouts provided by the instructor.
2. "Applications of the Classical Umbral Calculus", by Ira Gessel, Algebra univers., **49** (2003) 397–434, available from the instructor or online from http://arxiv.org/pdf/math/0108121v3.pdf
3. "generatingfunctionology", by Herbert S. Wilf, available as a pdf file from the instructor, or online at http://www.math.upenn.edu/ wilf/gfology2.pdf
4. The computer algebra system *Mathematica* will be used.

3 Undergraduate Projects

Project I This project requires a good deal of algebraic manipulations of series, and some computations. The use of Mathematica is recommended, and probably necessary for part (b).

(a) Find out how the telephone numbers t_n are related to the Hermite polynomials $H_n(u)$.

(b) Derive the formula in Theorem 4.5 of Gessel's paper using umbral methods, and check that the coefficients of x^n of the two sides agree for $n = 0, 1, 2, 3$.

(c) Use Theorem 4.5 to find the exponential generating function for t_{3n}.

(d) Draw the diagrams for all the possible telephone conversations in a network with 6 users.

Project II This project is more research oriented and partly open ended.

(a) Explain what the boustrophedon transform is, and how it originated.

(b) If b_n is the boustrophedon transform of a_n, use Umbral Calculus methods to express a_n in terms of b_n.

(c) Explore the boustrophedon transform of well-known sequences, such as the Bernoulli numbers, the Fibonacci numbers, etc.

(d) Look for other sequence transforms that are easily invertible, in the same way as the binomial or the boustrophedon transforms are.

Transform Methods and their Role in Classical Linear Algebra

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There are two transforms that have rather remarkable applications in the theory of operators on a finite dimensional vector space. They are the Laplace and the $\mathcal{Z}$ transforms. The Laplace transform is traditionally used in solving linear differential equations and systems. For systems the Laplace transform provides a rather simple approach for computing the matrix exponential of a $n \times n$ matrix A. From there we can prove the Cayley-Hamilton theorem and the describe the Jordan canonical form of A. For difference equations the $\mathbb{Z}$ transform replaces the role of the Laplace transform to give a rather simple method of computing A^k and again the Jordan canonical form of A. In this course we will explore all of the aforementioned ideas.

We will consider an $n \times n$ matrix A and discuss the Jordan Canonical form. For 2×2 matrices we can determine by brute force and in a case by case manner all of the various canonical matrices. Now for general $n \times n$ matrices the matter is a bit more difficult. We will define the Laplace and $\mathcal{Z}$- transforms and discuss their traditional uses. I will then apply these transform to matrix valued functions. For the Laplace transform we will compute e^{At} and for the $\mathbb{Z}$ transform we will compute A^k. Using these computations we will discuss how the Jordan canonical form 'falls out'.

Projects: The projects will be to apply the methods discussed to various types of matrices. For example, symmetric matrices and others. These projects will be more focussed in the second week of classes.

Texts: I will provide notes and a few relevant papers. If you have your differential equations book it may be helpful to bring that with you.