HYPERBOLIC GEOMETRY AND PARALLEL TRANSPORT IN $\mathbb{R}^2_+$

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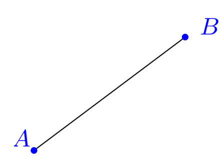
Abstract. We will examine the parallel transport of tangent vectors along a hyperbolic triangle containing sides of only geodesics in the upper half plane $\mathbb{R}^2_+$. We will also verify that the directed angle from the initial vector to the final vector is the negative area of the hyperbolic triangle.

1. Introduction

Geometry can be broken down into two types: Euclidean geometry and non-Euclidean geometry. Our paper is focusing on the parallel transport in hyperbolic geometry. Hyperbolic geometry is a non-Euclidean geometry, but before we can get into the characteristics of hyperbolic geometry, we must show the basics of Euclidean geometry so that the differences between the two will be apparent. The concept of Euclidean geometry can be satisfied using Euclid's five postulates which are listed and shown in the figures below:

(1) Any two points can be joined by a straight line.



(2) Any straight line segment can be extended indefinitely into a straight line.

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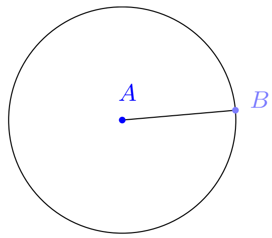
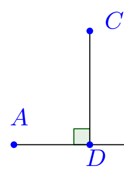
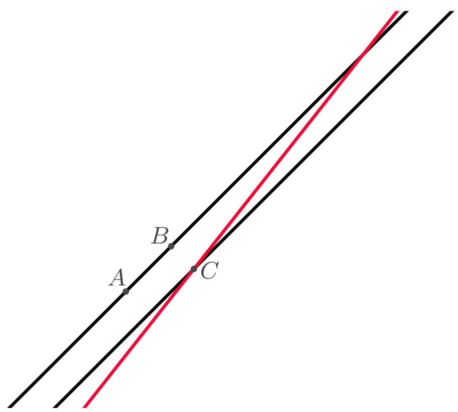
(3) Given any straight line segment, a circle can be drawn having the segment as a radius and one endpoint as center.



(4) All right angles are congruent.



(5) *Parallel Postulate :* Through any given point not on a line there passes exactly one line that is parallel to that line in the same plane.



The difference between Euclidean and non-Euclidean geometry is that the parallel postulate does not hold in non-Euclidean geometry. From this, we can now introduce what hyperbolic geometry is. Before we can define hyperbolic geometry, we need to lay out some background of hyperbolic geometry.

2. Background of Hyperbolic Geometry

Much of the history and development of Hyperbolic Geometry can be attributed to the mathematicians Gauss, Bolyai, and Lobachevsky. From its inception to now many mathematicians and physicists have found applications for hyperbolic geometry in complex variables, topology of two and three dimensional manifolds, finitely presented infinite groups, as well as physics and computer science.

2.1. **The Upper Half Plane.** One model of hyperbolic geometry is the upper-half plane which is defined by: $\mathbb{R}^2_+ = \{(x, y) \in \mathbb{R}^2 : y > 0\}$. This can also be interpreted in terms of complex numbers where $i = \sqrt{-1} : \mathbb{H}^2 = \{x + iy : x, y \in \mathbb{R}, y > 0\}$.

2.2. **The Hyperbolic Metric.** The Hyperbolic metric is considered a Riemmanian metric.

Definition 2.1. The assignment of an inner-product to each tangent space T_p i.e., $p \in \mathbb{R}^2_+ \mapsto \langle\ ,\ \rangle_p$ is a *Riemannian metric*, which can be represented by the matrix:

$$g(p) = \frac{1}{y_p^2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

where $p = (x_p, y_p)$.

2.3. **Geodesics.**

Definition 2.2. On a surface S, special curves called *geodesics* have the property that for any two points p and q sufficiently close, the length of a curve is less than or equal to any other curve joining p and q.

In hyperbolic geometry, a geodesic is either a vertical line or the arc of a semi-circle whose center is on the x-axis. Also, it can be referred to as a curve with an acceleration of zero. This differs from a geodesic in Euclidean geometry, in that the shortest distance between two points p and q is a straight line.

2.4. **Isometries.**

Definition 2.3. For Riemannian manifolds M and N, a function $f : M \to N$ is called an *isometry* if: $\langle u, v\rangle_p = \langle df_p(u), df_p(v)\rangle_{f(p)}$, for all $p \in M$, $u, v \in T_pM$.

Remark 2.4. A manifold is a space that locally looks like Euclidean space. A Riemannian manifold is a manifold with a Riemannian metric.

In other words, the distance between two points is preserved under f if f is an isometry.

2.5. **Affine Connections and Covariant Derivatives.**

Definition 2.5. Let R denote the set of vector fields of C^∞ on $\mathbb{R}^2_+$. An *affine connection* is a function where $\nabla : R \times R \to R$, denoted by (X,Y) $\mapsto \nabla_X Y$ satisfying:

(1) $\nabla_{f_1 X_1 + f_2 X_2} Z = f_1 \nabla_{X_1} Z + f_2 \nabla_{X_2}$.
(2) $\nabla_X (Y_1 + Y_2) = \nabla_X Y_1 + \nabla_X Y_2$.
(3) $\nabla_X (fY) = X(f)Y + f \nabla_X Y$.

where f, $f_1,\ f_2 \in C^\infty(\mathbb{R}^2_+)$.

Remark 2.6. $C^\infty(\mathbb{R}^2_+)$ denotes the smooth functions in $\mathbb{R}^2_+$, so all partial derivatives exist.

Theorem 2.7. *Let V be a vector field along the differentiable curve c and let $\frac{DV}{dt}$ be the covariant derivative of V along c, then the function $V \mapsto \frac{DV}{dt}$ is unique provided*

(1) $\frac{D(V_1+V_2)}{dt} = \frac{DV_1}{dt} + \frac{DV_2}{dt}$, *where V_1, V_2 are vector fields along the curve γ,*
(2) $\frac{D(fV)}{dt} = f'(t)V + f\frac{DV}{dt}$, *whenever $t \mapsto f(t) \in \mathbb{R}$ is C^∞, and*
(3) *If $V(t) = Y(\gamma(t))$ for some vector field $Y \in \mathcal{D}$, then $\frac{DV}{dt}\nabla_{\dot\gamma(t)}(Y)$.*

Definition 2.8. The coefficients $\Gamma^m_{i,j}$ are called *Christoffel symbols.* In $\mathbb{R}^2_+$, the Christoffel symbols satisfy:

$$-\Gamma^2_{1,1} = \Gamma^1_{1,2} = \Gamma^1_{2,1} = \Gamma^2_{2,2} = -\frac{1}{y}$$

The covariant derivative will be helpful in allowing us to take the derivative of vector fields instead of just functions.

Lemma 2.9. *The affine connection ∇ on $\mathbb{R}^2_+$ satisfies*

(1) $\nabla_{\partial_x}(\frac{\partial}{\partial x}) = \frac{1}{y}\frac{\partial}{\partial y}$

(2) $\nabla_{\partial_x}(\frac{\partial}{\partial y}) = -\frac{1}{y}\frac{\partial}{\partial x}$

(3) $\nabla_{\partial_y}(\frac{\partial}{\partial x}) = -\frac{1}{y}\frac{\partial}{\partial x}$

(4) $\nabla_{\partial_y}(\frac{\partial}{\partial y}) = -\frac{1}{y}\frac{\partial}{\partial y}$

3. Parallel Transport

To connect the concepts of parallel lines and interior angles of a triangle, we introduce the notion of a *parallel transport*.

Definition 3.1. Let v and w be two vectors in $\mathbb{R}^2$. Let p and q be two points on curve L. The initial points of v and w are p and q, respectively. We say the *parallel transport* of vector v is vector w provided v and w form the same angles with curve L, and v and w have the same lengths. Below in figure 1 is a picture of a parallel transport:

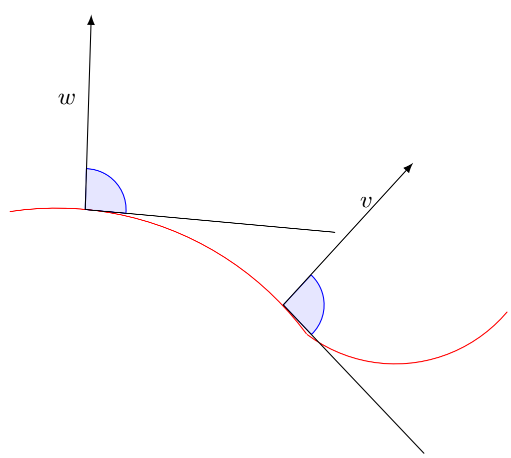


Figure 1. Parallel Transport

We will now show the derivation of a system of differential equations that can be used to find the parallel vector fields with respect to curves.

Proof. Given $V = (f(t), g(t))$ on the vector field $\gamma(t) = (\gamma_1(t), \gamma_2(t))$. Find when $\frac{DV}{dt} = 0$.

$$\frac{DV}{dt} = \frac{D\left(f(t)\frac{d}{dx} + g(t)\frac{d}{dy}\right)}{dt}$$

$$= \frac{d}{dt}(f(t))\frac{d}{dx} + f(t)\frac{D}{dt}\frac{d}{dx} + \frac{d}{dt}(g(t))\frac{d}{dy} + g(t)\frac{D}{dt}\frac{d}{dy}$$

$$= f'(t)\frac{d}{dx} + f(t)\nabla_{\gamma'(t)}\frac{d}{dx} + g'(t)\frac{d}{dy} + g(t)\nabla_{\gamma'(t)}\frac{d}{dy}$$

Now using the fact that $\gamma'(t) = \gamma_1'(t) + \gamma_2'(t)$ we get:

$$f'(t)\frac{d}{dx} + f(t)\nabla_{\gamma_1'(t)+\gamma_2'(t)}\frac{d}{dx} + g'(t)\frac{d}{dy} + g(t)\nabla_{\gamma_1'(t)+\gamma_2(t)}\frac{d}{dy}$$

We know that $\nabla_{\gamma_1'(t)+\gamma_2'(t)} \times Z$ can be rewritten as $\nabla_{\gamma_1'(t)} \times Z + \nabla_{\gamma_2'(t)} \times Z$. Therefore we get:

$$\begin{aligned}
&f'(t)\frac{d}{dx} + f(t)\left(\nabla_{\gamma_1'(t)}\frac{d}{dx} + \nabla_{\gamma_2'(t)}\frac{d}{dx}\right) \\
&+g'(t)\frac{d}{dy} + g(t)\left(\nabla_{\gamma_1'(t)}\frac{d}{dy} + \nabla_{\gamma_2'(t)}\frac{d}{dy}\right) \\
&= f'(t)\frac{d}{dx} + f(t)\nabla_{\gamma_1'(t)}\frac{d}{dx} + f(t)\nabla_{\gamma_2'(t)}\frac{d}{dx} \\
&+g'(t)\frac{d}{dy} + g(t)\nabla_{\gamma_1'(t)}\frac{d}{dy} + g(t)\nabla_{\gamma_2'(t)}\frac{d}{dy}
\end{aligned}$$

Now using to fact that $\nabla_{p(t)dx} = p(t)\nabla_{dx}$ and knowing that $\gamma_1'(t)$ is the x part of $\gamma(t)$ and $\gamma_2'(t)$ in the y part, we get:

$$\begin{aligned}
&f'(t)\frac{d}{dx} + f(t)\gamma_1'(t)\nabla_{dx}\frac{d}{dx} + f(t)\gamma_2'(t)\nabla_{dy}\frac{d}{dx} \\
&g'(t)\frac{d}{dy} + g(t)\gamma_1'(t)\nabla_{dx}\frac{d}{dy} + g(t)\gamma_2'(t)\nabla_{dy}\frac{d}{dy}
\end{aligned}$$

$$f'(t)\frac{d}{dx}+f(t)\frac{\gamma_1'(t)}{y}\frac{d}{dy}-f(t)\frac{\gamma_2'(t)}{y}\frac{d}{dx}+g'(t)\frac{d}{dy}-g(t)\frac{\gamma_1'(t)}{y}\frac{d}{dx}-g(t)\frac{\gamma_2'(t)}{y}\frac{d}{dy}$$

Now to equal 0 we need the x and y parts to both equal 0 and that the y in the previous equation is referring to the y coordinate of the vector field $\gamma(t)$, so we get the set of equations:

$$\begin{aligned}
f'(t)\frac{d}{dx} - \frac{\gamma_2'(t)}{\gamma_2(t)}f(t)\frac{d}{dx} - \frac{\gamma_1'(t)}{\gamma_2(t)}g(t)\frac{d}{dx} = 0. \\
g'(t)\frac{d}{dy} + \frac{\gamma_1'(t)}{\gamma_2(t)}f(t)\frac{d}{dy} - \frac{\gamma_2'(t)}{\gamma_2(t)}g(t)\frac{d}{dy} = 0.
\end{aligned}$$

Now we drop the $\frac{d}{dx}$ and $\frac{d}{dy}$ since we grouped them by the symbols and we solve for $f'(t)$ and $g'(t)$.

$$\begin{aligned}
f'(t) &= \frac{\gamma_2'(t)}{\gamma_2(t)}f(t) + \frac{\gamma_1'(t)}{\gamma_2(t)}g(t). \\
g'(t) &= -\frac{\gamma_1'(t)}{\gamma_2(t)}f(t) + \frac{\gamma_2'(t)}{\gamma_2(t)}g(t).
\end{aligned}$$

$\square$

Example 3.2. We will calculate the parallel transport of a vector V_0 with angle θ between V_0 and the y-axis along a rectangle. The curve γ is given by $\gamma_1(t,1)$ and $\gamma_2(t,2)$ along the horizontal lines and $\eta_{\frac{\pi}{2}}(\frac{\pi}{2}, e^t)$ and $\eta_0(0, e^t)$. $V(t) = (b\sin(\theta + \frac{t}{b}), b\cos(\theta+\frac{t}{b}))$ and $W(t) = (e^t \sin\theta, e^t\cos\theta)$ are the vector fields, and c is a simple closed rectangular curve from $(0,1)$ to $(\frac{\pi}{2}, 1)$ to $(\frac{\pi}{2}, 2)$ to $(0,2)$ to $(0,1)$.

We start by checking that V and W are parallel along γ and η by showing that:

$$\frac{DW(t)}{dt} = 0 = \frac{DV(t)}{dt}.$$

$$\begin{aligned}
\frac{DW}{dt} &= \frac{d}{dt}(b\sin(\theta+\frac{t}{b}))\frac{d}{dx}\bigg|_{(t,b)} + (b\sin(\theta+\frac{t}{b}))\frac{D}{dt}(\frac{d}{dx}\bigg|_{(t,b)}) \\
&+ \frac{d}{dt}(b\cos(\theta+\frac{t}{b}))\frac{d}{dy}\bigg|_{(t,b)} + (b\cos(\theta+\frac{t}{b}))\frac{D}{dt}(\frac{d}{dy}\bigg|_{(t,b)}) \\
&= \frac{1}{b}b\cos(\theta+\frac{t}{b})\frac{d}{dx} + b\sin(\theta+\frac{t}{b})\nabla_{h_b(t)}\frac{d}{dx} \\
&+ (-\frac{1}{b})b\sin(\theta+\frac{t}{b})\frac{d}{dy} + b\cos(\theta+\frac{t}{b})\nabla_{h_b(t)}\frac{d}{dy} \\
&= \cos(\theta+\frac{t}{b})\frac{d}{dx} + b\sin(\theta+\frac{t}{b})\nabla_{dx}\frac{d}{dx} - \sin(\theta+\frac{t}{b})\frac{d}{dy} + b\cos(\theta+\frac{t}{b})\nabla_{dx}\frac{d}{dy} \\
&= \cos(\theta+\frac{t}{b})\frac{d}{dx} + b\sin(\theta+\frac{t}{b})\frac{1}{y}\frac{d}{dy} - \sin(\theta+\frac{t}{b})\frac{d}{dy} + b\cos(\theta+\frac{t}{b})(-\frac{1}{y})\frac{d}{dx} \\
&= \cos(\theta+\frac{t}{b})\frac{d}{dx} + \sin(\theta+\frac{t}{b})\frac{d}{dy} - \sin(\theta+\frac{t}{b})\frac{d}{dy} - \cos(\theta+\frac{t}{b})\frac{d}{dx} \\
&= 0
\end{aligned}$$

Also,

$$\begin{aligned}
\frac{DV}{dt}\bigg|_{(a,e^t)} &= \frac{d}{dt}(e^t\sin\theta)\frac{\partial}{\partial x}\bigg|_{(a,e^t)} + e^t\sin\theta\frac{D}{dt}\left(\frac{\partial}{\partial x}\bigg|_{(a,e^t)}\right) \\
&+ \frac{d}{dt}(e^t cos\theta)\frac{\partial}{\partial y}\bigg|_{(a,e^t)} + e^t cos\theta\frac{D}{dt}\left(\frac{\partial}{\partial y}\bigg|_{(a,e^t)}\right).
\end{aligned}$$

By part 3 of 2.7,

$$\begin{aligned}\frac{DV}{dt}\bigg|_{(a,e^t)} &= e^t\sin\theta\frac{\partial}{\partial x}\bigg|_{(a,e^t)} + e^t\sin\theta\nabla_{\dot\gamma_a(t)}\left(\frac{\partial}{\partial x}\right)\\ &+ e^t\cos\theta\frac{\partial}{\partial y}\bigg|_{(a,e^t)} + e^t\cos\theta\nabla_{\dot\gamma_{(a,t)}}\left(\frac{\partial}{\partial y}\right).\end{aligned}$$

Notice, if $Y(x,y) = y\frac{\partial}{\partial y}$ is a vector field on $\mathbb{R}^2_+$,
then $Y(\gamma_a(t)) = e^t\frac{\partial}{\partial y}|_{(,e^t)} = \dot\gamma_a(t)$.

$$\begin{aligned}\frac{DV}{dt}\bigg|_{(a,e^t)} &= e^t\sin\theta\frac{\partial}{\partial x}\bigg|_{(a,e^t)} + e^t\sin\theta\left(\nabla_Y\left(\frac{\partial}{\partial x}\right)\right)(\gamma_a(t))\\ &+ e^t\cos\theta\frac{\partial}{\partial y}\bigg|_{(a,e^t)} + e^t\cos\theta\left(\nabla_Y\left(\frac{\partial}{\partial y}\right)\right)(\gamma_a(t)).\\ &= e^t\sin\theta\frac{\partial}{\partial x}\bigg|_{(a,e^t)} + e^t\sin\theta\left(y\nabla_{\partial/\partial y}\left(\frac{\partial}{\partial x}\right)\right)(\gamma_a(t))\\ &+ e^t\cos\theta\frac{\partial}{\partial y}\bigg|_{(a,e^t)} + e^t\cos\theta\left(y\nabla_{\partial/\partial y}\left(\frac{\partial}{\partial y}\right)\right)(\gamma_a(t)).\end{aligned}$$

Applying the Christoffel symbols in 2.8, we get:

$$\begin{aligned}\frac{DV}{dt}\bigg|_{(a,e^t)} &= e^t\sin\theta\frac{\partial}{\partial x}\bigg|_{(a,e^t)} + e^t\sin\theta\left(-\frac{\partial}{\partial x}\right)(\gamma_a(t))\\ &+ e^t\cos\theta\frac{\partial}{\partial y}\bigg|_{(a,e^t)} + e^t\cos\theta\left(-\frac{\partial}{\partial y}\right)(\gamma_a(t))\\ &= 0 \in T_{(a,e^t)}.\end{aligned}$$

Remark 3.3. $T_{(a,e^t)}$ is the set of all tangent vectors with initial points (a, e^t).

Since the fields are indeed parallel, we can calculate a parallel transport along a closed curve, a rectangle. The curves and vector fields were defined above. Refer to Figure 2 to see how the vectors are parallel transported about the rectangle. The blue vectors on the graph are the tangent lines to the rectangle, and the red vectors on the graph represent the transported vectors. The starting position vector is labeled:

$$v_0 = (\sin\theta, \cos\theta).$$

We start by evaluating $\gamma_1(t)$ to get our initial vector. The tangent vector is $(0,1)$. We let θ represent the angle from the tangent vector to

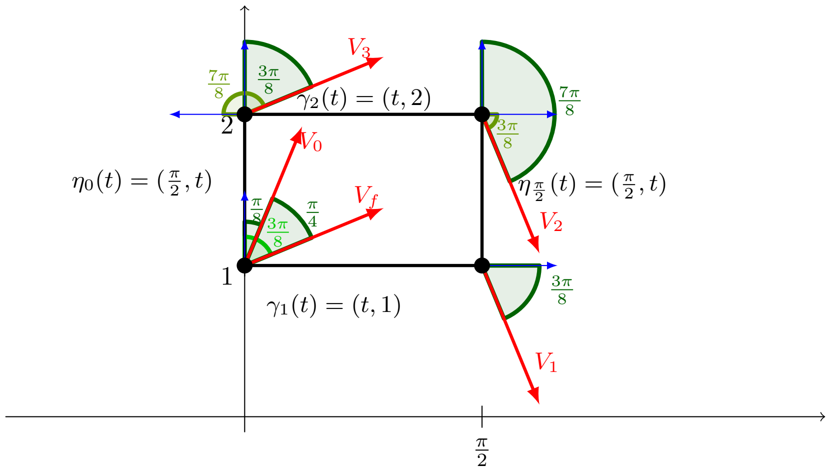


FIGURE 2. Parallel Transport Around a Rectangle

the initial vector. After moving along γ_1 the tangent vector becomes:

$$\eta_{\frac{\pi}{2}}(0) = \left(\frac{\pi}{2}, 1\right).$$

Since we are transporting from $(\frac{\pi}{2}, 1)$ to $(\frac{\pi}{2}, 2)$, we need to solve for t when $e^t = 2$, thus $t = \ln(2)$. So this gets us the point $(\frac{\pi}{2}, 2)$, by maintaining the angle $(\sin(\theta + \frac{\pi}{2}), \cos(\theta + \frac{\pi}{2}))$. We evaluate:

$$V(\ln 2) = (2\sin(\theta + \frac{\pi}{2}), 2\cos(\theta + \frac{\pi}{2})).$$

Our new vector to be transported across the horizontal line is

$$V_3 = (2\sin(\alpha + \frac{t}{2}), 2\cos(\alpha + \frac{t}{2}))$$

and $W(0)$ is now $(2\sin\alpha, 2\cos\alpha)$ in which α is the angle between the tangent vector and V_3. Next we evaluate $W(t)$ at $t = \frac{\pi}{2}$ to get the vector

$$v_3 = (2\sin(\alpha + \frac{\pi}{4}), 2\cos(\alpha + \frac{\pi}{4})).$$

By solving for α in the equation,

$$\alpha + \frac{\pi}{4} = \theta + \frac{\pi}{2},$$

we get

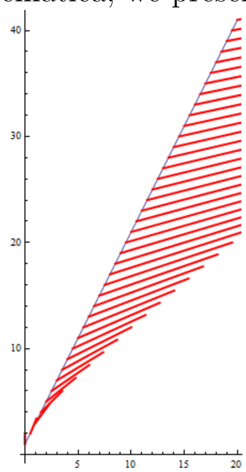
$$\alpha = \theta + \frac{\pi}{4}.$$

Here we are transporting along a vertical line which is a geodesic, so the angle remains the same. The angle of our final vector once we return to the starting point is $\frac{\pi}{4}$.

Given the curve $c(t) = (t, mt+b)$, we used Mathematica in collaboration with the previously derived system of differential equations to find the equation for the parallel vector field, $V(t) = (V_1(t), V_2(t))$ where:

$$\begin{aligned} V_1(t) &= e^{-\frac{t}{b+\mathrm{mt}}}\left(e^{\frac{t}{b+\mathrm{mt}}} - 1\right), \\ V_2(t) &= e^{-\frac{t}{b+\mathrm{mt}}} \end{aligned}$$

Using Mathematica, we present the parallel transport of the line



$y = 2t + 1$: Next, we will show a parallel transport along a hyperbolic triangle with two points on the y-axis.

Example 3.4. We will show that by parallel transporting a vector about a hyperbolic triangle, we can find the defect angle, or the difference in angle between the initial vector and transported vector.

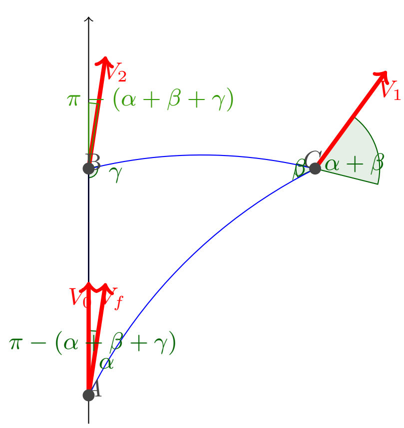


FIGURE 3. Parallel Transport About Hyperbolic Triangle

By parallel transporting our original vector about the hyperbolic triangle, we have shown that the defect angle is equal to π - $(\alpha + \beta + \gamma)$. In both the rectangle and the hyperbolic triangle, there seems to be a correlation between the areas of the figures and the defect angle after the transport. In the next section, we will confirm this speculation.

4. Area Correlation to Parallel Transport of Vectors

To find the area of the rectangle, we use the following integral:

$$\begin{aligned} A &= \int_0^{\frac{\pi}{2}} \int_1^2 \frac{dydx}{y^2} \\ &= \int_0^{\frac{\pi}{2}} (\frac{-1}{y}\Big|_1^2)\, dx \\ &= \int_0^{\frac{\pi}{2}} \frac{1}{2}\, dx \\ &= \frac{1}{2}x\Big|_0^{\frac{\pi}{2}}\, dx \\ &= \frac{\pi}{4}. \end{aligned}$$

Before we can prove this using the hyperbolic triangle, we must first show that the maximum area of a hyperbolic triangle is π. To do so, we will verify that the area of a hyperbolic triangle with angles α, β, and $\gamma = 0$ satisfies the following equation:

$$A = \int_{-\cos\alpha}^{\cos\beta} \int_{\sqrt{1-x^2}}^{\infty} \frac{dydx}{y^2} = \pi - (\alpha + \beta)$$

Intuitively, we see that it is given by figure 4:

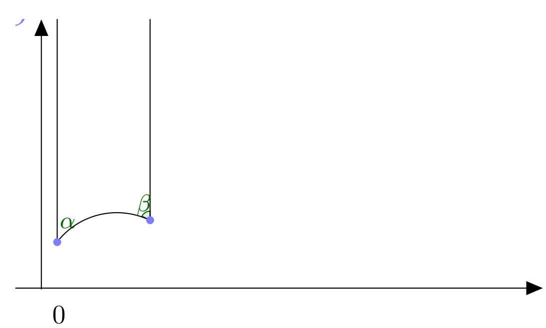


Figure 4. Area of $\triangle AB\infty$

Theorem 4.1. *The angle sum of a hyperbolic triangle is less than π.*

Proof. To prove this, we will show that the angles of a hyperbolic triangle α, β, and $\gamma = 0$ with points at A, B, and ∞ satisfy the following equation:
We calculate:

$$A = \int_{-\cos\alpha}^{\cos\beta} \int_{\sqrt{1-x^2}}^{\infty} \frac{dydx}{y^2} = \pi - (\alpha + \beta).$$

$$\begin{aligned}
&= \int_{-\cos\alpha}^{\cos\beta} \int_{\sqrt{1-x^2}}^{\infty} \frac{dydx}{y^2} \\
&= \int_{-\cos\alpha}^{\cos\beta} \frac{-1}{y}\Big|_{\sqrt{1-x^2}}^{\infty} dx \\
&= \int_{-\cos\alpha}^{\cos\beta} \frac{1}{\sqrt{1-x^2}}\, dx \\
&= \arcsin(x)\Big|_{-\cos\alpha}^{\cos\beta} \\
&= \arcsin(\sin(\frac{\pi}{2} - \beta)) - \arcsin(-\sin(\frac{\pi}{2} - \alpha)) \\
&= (\frac{\pi}{2} - \beta) + (\frac{\pi}{2} - \alpha).
\end{aligned}$$

Thus, we have

$$A = \int_{-\cos\alpha}^{\cos\beta} \int_{\sqrt{1-x^2}}^{\infty} \frac{dydx}{y^2} = \pi - (\alpha + \beta).$$

Since α,β>0, it is clear that the sum of the angles of a hyperbolic triangle is less than π.

□

We will now generalize this result to find the area of any hyperbolic triangle.

Theorem 4.2. *The area of a hyperbolic triangle with angles α, β, and γ is $A(\triangle ABC) = \pi$ - $(\alpha + \beta + \gamma)$.*

Proof. We start by letting our original $\beta = \beta_1 + \beta_2$.
We will find the area of $\triangle ABC$ by finding the area of $\triangle AB\infty$ and subtracting $\triangle BC\infty$, which we have already calculated, from that.

$$
\begin{aligned}
A &= \int_{-\cos\alpha}^{\cos\beta}\int_{\sqrt{1-x^2}}^{\infty}\frac{dydx}{y^2} - \int_{-\cos(\pi-\gamma)}^{\cos\beta_2}\int_{\sqrt{1-x^2}}^{\infty}\frac{dydx}{y^2} \\
&= \int_{-\cos\alpha}^{\cos\beta}\frac{-1}{y}\Big|_{\sqrt{1-x^2}}^{\infty}dx - \int_{-\cos(\pi-\gamma)}^{\cos\beta_2}\frac{-1}{y}\Big|_{\sqrt{1-x^2}}^{\infty}dx \\
&= \int_{-\cos\alpha}^{\cos\beta}\frac{1}{\sqrt{1-x^2}}dx - \int_{-\cos(\pi-\gamma)}^{\cos\beta_2}\frac{1}{\sqrt{1-x^2}}dx \\
&= \arcsin(x)\Big|_{-\cos\alpha}^{\cos\beta} - \arcsin(x)\Big|_{-\cos(\pi-\gamma)}^{\cos\beta_2}dx \\
&= \arcsin(\sin(\frac{\pi}{2}-\beta)) - \arcsin(-\sin(\frac{\pi}{2}-\alpha)) \\
&- (\arcsin(\sin(\frac{\pi}{2}-\gamma)) - \arcsin(-\sin(\frac{\pi}{2}-\gamma))) \\
&= (\frac{\pi}{2}-\beta) + (\frac{\pi}{2}-\alpha) - ((\frac{\pi}{2}-\beta_2) + (\frac{\pi}{2}-\gamma)) \\
&= \pi - (\alpha+\beta) - (\pi - (\gamma+\beta_2)) \\
&= \pi - (\alpha+\beta_1+\beta_2) - (\pi - (\gamma+\beta_2)) \\
&= \pi - (\alpha+\beta+\gamma)
\end{aligned}
$$

Therefore, we have shown that the area of any hyperbolic triangle with angles α, β, and γ is π - (α + β + γ). □

Both of the figures we used to show the parallel transports and areas had at least two points on the y-axis, but we also wanted to know how we could parallel transport along a figure that wasn't on the y-axis. We found out that we could use a Möbius Transformation to map the figure to the y-axis to be able to parallel transport the vectors.

5. Möbius Transformation of an Arbitrary Hyperbolic Triangle

Given a triangle $\triangle ABC$ where $A = (X_A, Y_A)$, $B = (X_B, Y_B)$, and $C = (X_C, Y_C)$, we want to find the Möbius transformation of the arbitrary hyperbolic triangle that translates the points of the triangle to the points $(0, 1)$,$(0, Y_0)$, and(A, B) respectively where $A, B > 0$.

First we want to find a matrix NM for which $f_{NM}(A) = i$. Remember that A, B, and C can be written as $X_A + Y_A i$, $X_B + Y_B i$, and $X_C + Y_C i$ respectively. Let matrix $M = \begin{bmatrix} 1 & -X_A \\ 0 & 1 \end{bmatrix}$. This gives us the following:

$$f_M(A) = \frac{1(X_A + Y_A i) + (-X_A)}{0(X_A + Y_A i) + 1} = Y_A i.$$

In order to cancel the Y_A of $f_M(A)$, let $N = \begin{bmatrix} 1 & 0 \\ 0 & Y_A \end{bmatrix}$.

$$f_{NM}(B) = \frac{X_B - X_A + Y_B i}{Y_A}$$

which, written as a coordinate, would look like $(\frac{X_B - X_A}{Y_A}, \frac{Y_B}{Y_A})$. We use a rotation matrix, K, which is always of the form $\begin{bmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix}$, to move the $f_{NM}(B)$ to the y-axis as seen in Figure 5.

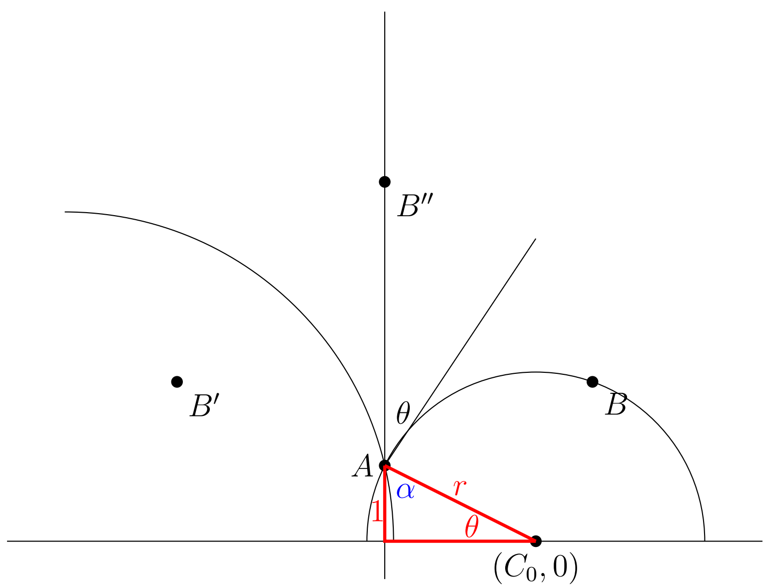


FIGURE 5. Möbius transformation

Now we need to find θ, the angle between tangent line of the semi-circle through the points A and B at A and the y-axis.

We notice that θ is also the angle between the line from the center of the circle through the points A and B to A, which is the point $(0, 1)$, and the x-axis. Therefore $\tan\theta = \frac{1}{C_0}$. C_0 can be found with the equation:

$$C_0 = \frac{(\frac{y_B}{y_A})^2 + (\frac{x_B - x_A}{y_A})^2 - 1}{2(\frac{x_B - x_A}{y_A})}$$

$$\text{So } \frac{1}{C_0} = \frac{2(\frac{x_B - x_A}{y_A})}{(\frac{y_B}{y_A})^2 + (\frac{x_B - x_A}{y_A})^2 - 1}$$

$$\text{Therefore } \theta = \arctan\left(\frac{2(\frac{x_B - x_A}{y_A})}{(\frac{y_B}{y_A})^2 + (\frac{x_B - x_A}{y_A})^2 - 1}\right)$$

From this we get the matrix K:

$$K = \begin{bmatrix} \cos\left(\arctan\left(\frac{(\frac{x_B - x_A}{y_A})}{(\frac{y_B}{y_A})^2 + (\frac{x_B - x_A}{y_A})^2 - 1}\right)\right) & -\sin\left(\arctan\left(\frac{(\frac{x_B - x_A}{y_A})}{(\frac{y_B}{y_A})^2 + (\frac{x_B - x_A}{y_A})^2 - 1}\right)\right) \\ \sin\left(\arctan\left(\frac{(\frac{x_B - x_A}{y_A})}{(\frac{y_B}{y_A})^2 + (\frac{x_B - x_A}{y_A})^2 - 1}\right)\right) & \cos\left(\arctan\left(\frac{(\frac{x_B - x_A}{y_A})}{(\frac{y_B}{y_A})^2 + (\frac{x_B - x_A}{y_A})^2 - 1}\right)\right) \end{bmatrix}$$

So, now the equations for the points of the transformed $\triangle$ are $A' = f_{KNM}(A)$, $B' = f_{KNM}(B)$, and $C' = f_{KNM}(C)$.

6. Future studies

Some future studies on the parallel transport might be to look at it in different models of hyperbolic geometry such as the Poincaré disk, and the hyperboloid model.

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