LINEAR GEOMETRIC CONSTRUCTIONS

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Abstract. We will discuss and give examples of two ways to create geometric constructions and the tools needed to produce these constructions. One way uses a straightedge and compass, while the other uses a straightedge with two notches and a compass, otherwise known as the neusis construction. Also, we will discuss those numbers that are constructible using these two ways and those that we cannot currently construct.

1. introduction

A Geometric Construction is a construction of lengths, angles, and geometric figures using only a straightedge and compass. The ancient Greeks' usage of the phrase "to construct" is similar to the way modern mathematicians "Show things exist". There are two types of Geometric constructions: there are those that can be constructed using only a straightedge and compass and those that can be constructed using a straightedge with two notches and a compass. Some famous mathematicians who made great strives in Geometric Constructions include Plato, Euclid, Pythagoras, and Thales.

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2. Straight edge and compass constructions

We assume that two distinct points are constructible. We will call them O, the origin, and (1,0). The line through two constructible points is a *constructible line.* Given constructible points A and B, we can construct a *constructible circle* with center either A or B and radius $\overline{AB}$.

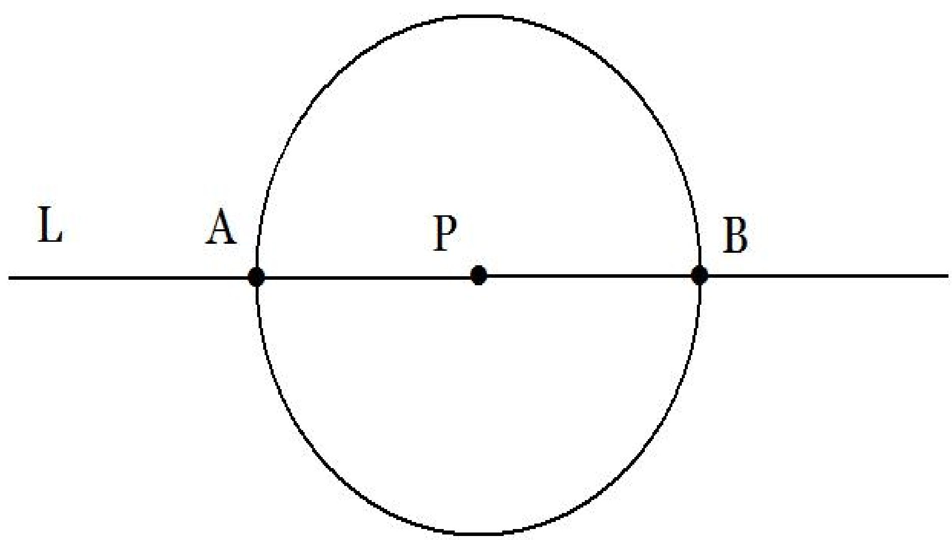
Definition 2.1. A point is constructible if it is the intersection point of two constructed lines or circles or a combination of both.

2.1. **Basic constructions.**

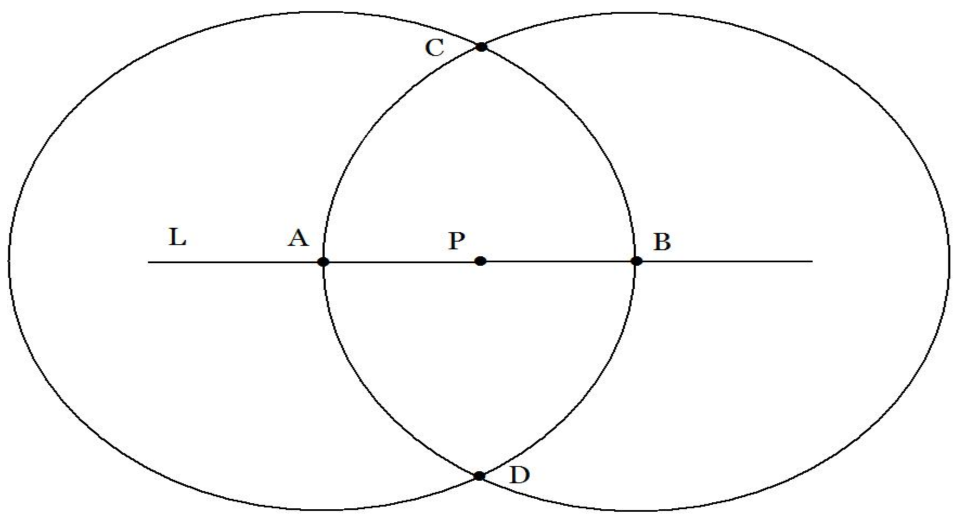
Theorem 2.2 (**Raise a perpendicular**). *Given a line L and a point P on the line, we can draw a line perpendicular to L that passes through P.*

Construction:

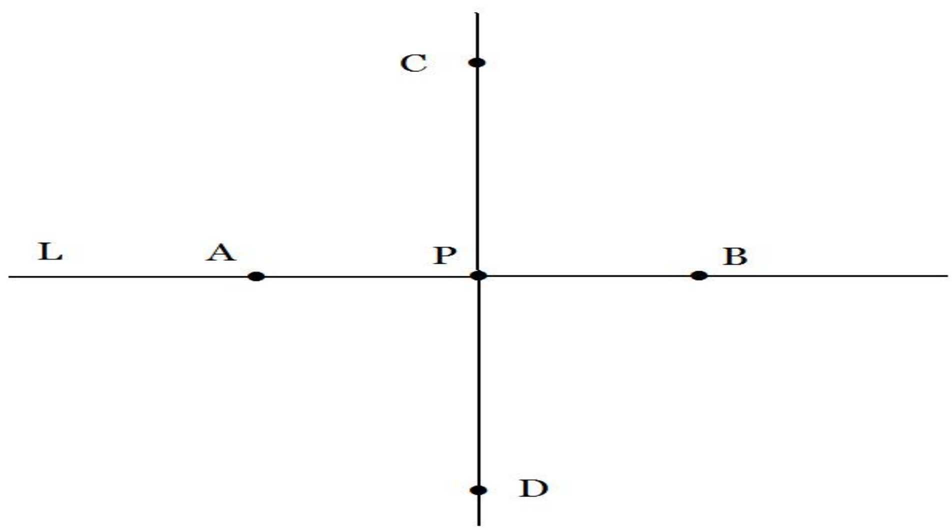
(1) First we draw a circle with center at point P and get points A and B where the perimeter of the circle intersects with the line L.



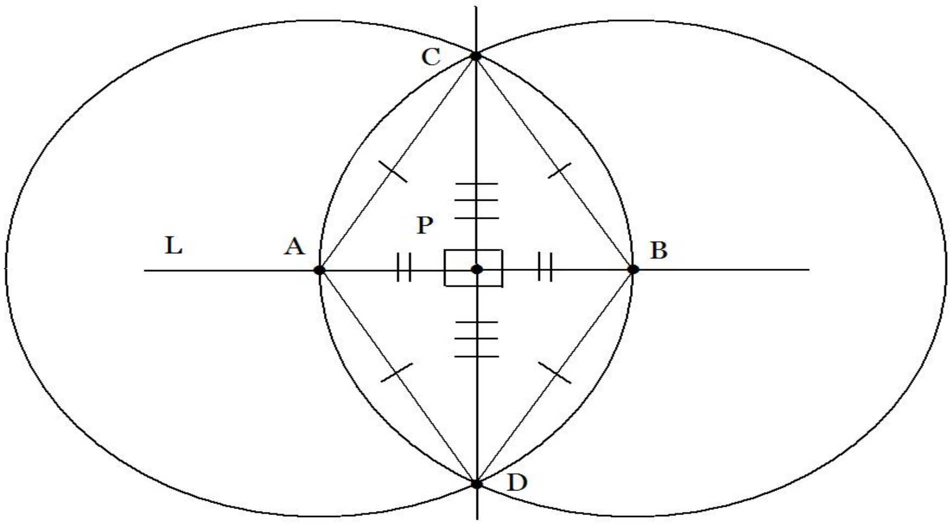
(2) Next we draw two circles with radius $\overline{AB}$ and centers at A and B. These two circles intersect at points C and D.



(3) Then we use a straight edge to connect points C and D. We claim the line segment $\overline{CD}$ runs through point P and is perpendicular to the line L.



Proof. We use a straight edge to draw the four equal segments $\overline{AC}$,$\overline{CB}$,$\overline{BD}$, and $\overline{DA}$. We know these segments are equal because they are all radii two congruent circles. These segments produce the rhombus $ACBD$. By properties of the rhombus, the two diagonals $\overline{CD}$ and $\overline{AB}$ are perpendicular to each other and split each other in half, producing the segments $\overline{AP}$, $\overline{PA}$, $\overline{PC}$, and $\overline{PD}$.



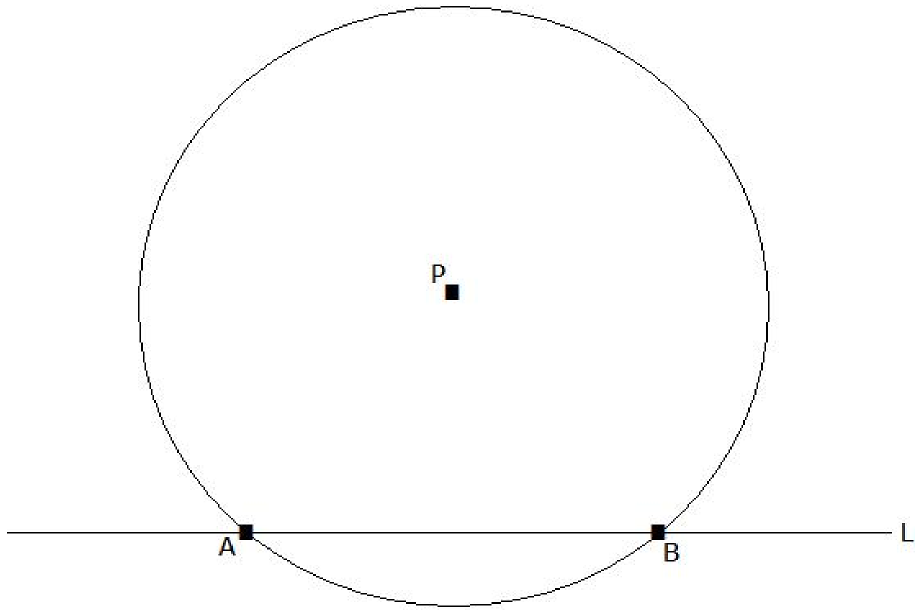
□

Theorem 2.3 (**Drop a perpendicular**). *Given a line L and a point P not on L. We can construct a line perpendicular to L that passes through P*

Construction:

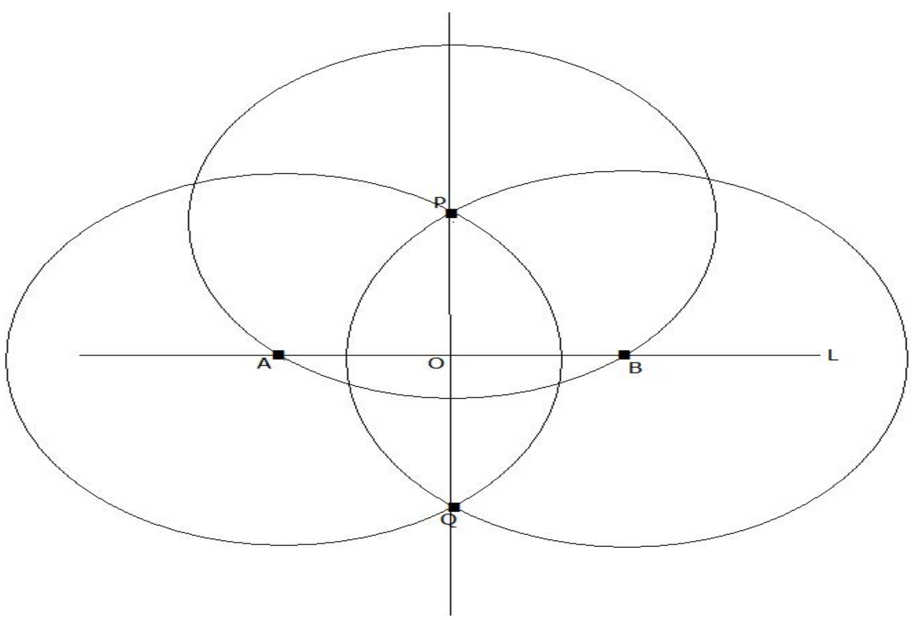
(1) Pick a point A somewhere on the line L.

(2) Circle PA to get point B. If the circle PA is tangent to the line L, the segment $\overline{PA}$ is perpendicular to L



(3) Circle AP and BP to get point Q at the intersection of the two circles.

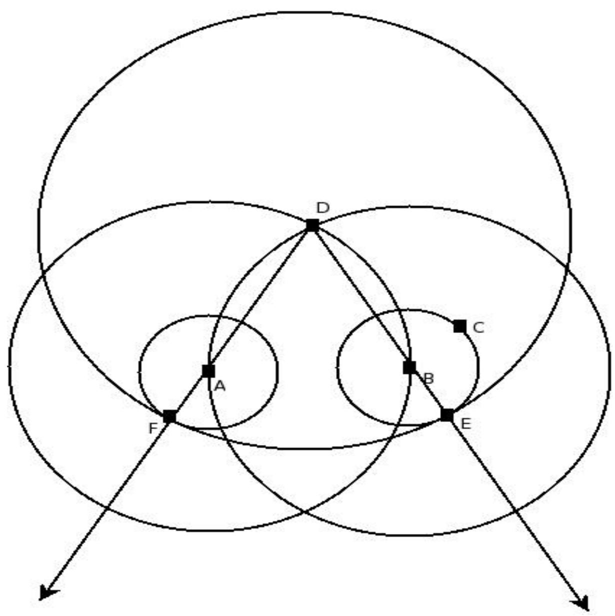
(4) Use a straight edge to draw the line $\overline{PQ}$. The line $\overline{PQ}$ is perpendicular to L.



Proof. After we draw the line segments $\overline{PB}$, $\overline{PA}$, $\overline{AQ}$, and $\overline{BQ}$ we follow the same proof as the one used for raising a perpendicular. □

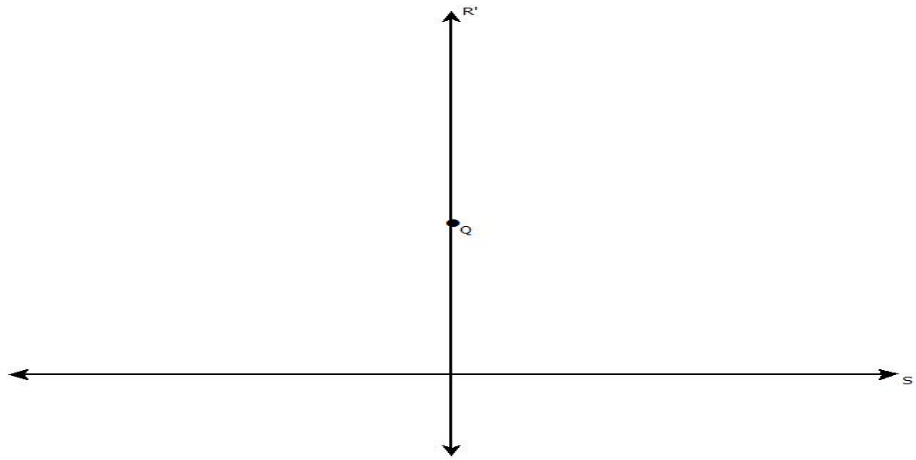
Theorem 2.4 (**Rusty Compass Theorem**). *Given a line segment ($\overline{AB}$) and a point C, we can draw the circle with center C and radius the length of $l(\overline{AB})$.*

Proof. First, we draw a circle centered at A with point B on the circle and also draw a circle centered at B with point A on the circle. The intersection of these two circles will be our point D and form △ABD. Extend the line $\overline{BD}$ passed the intersection of circle BE, creating point E. Next, draw circle DE. Like the past step, Extend the line $\overline{AD}$ until it intersects the circle DE. Name this intersection, point F. Now, construct the circle AF. Since E is on the circle BC, $l(\overline{BE}) = l(\overline{BC})$. Since $\triangle ADB$ is an equilateral triangle, $l(\overline{DA}) = l(\overline{DB})$. Since E and F are on the circle D, $l(\overline{DE}) = l(\overline{DF})$. Therefore, $l(\overline{AF}) = l(\overline{BE})$ and $l(\overline{AF}) = l(\overline{BC})$. □

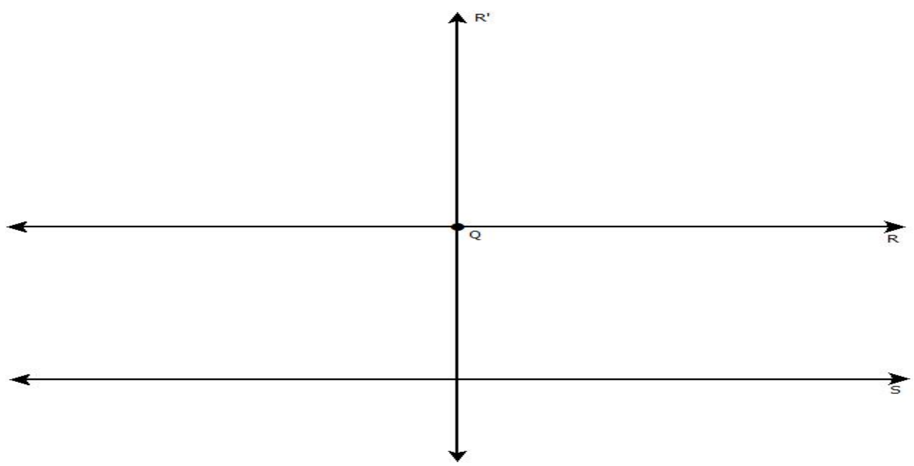


Theorem 2.5 (**Constructing a parallel line**). *Given a line S and a point Q not on the line S. We can construct a line R through the point Q that is parallel to the line S.*

(1) Using a straightedge and compass, construct a perpendicular line to S through the point Q. Name R'.



(2) Using a straightedge and compass, construct a perpendicular line to R' through the point Q. Name R.

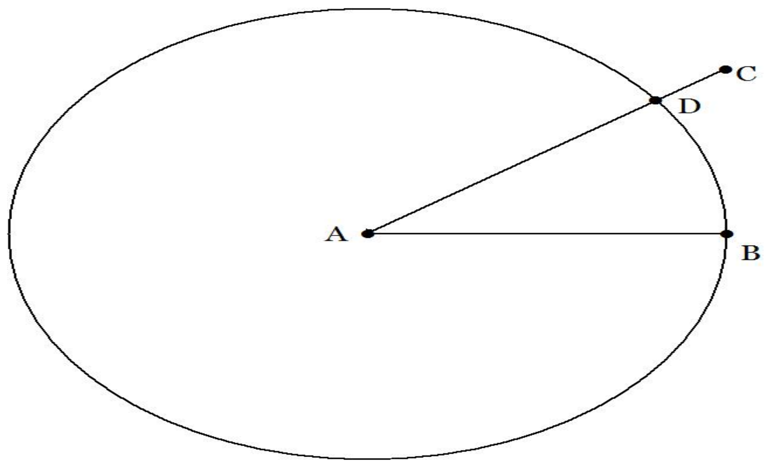


Proof. Line R and line S are both perpendicular to line R', and line R and line S are not the same line since Q is a point that is on R but not S. Therefore line R and line S are parallel. □

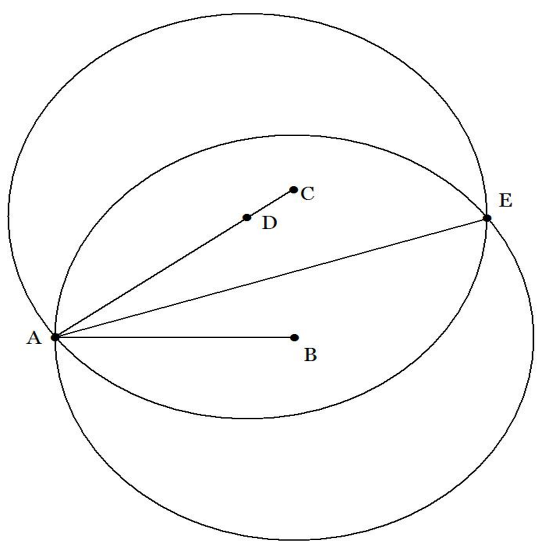
Theorem 2.6 (Bisecting the angle). *Given an angle BAC, we can bisect the angle.*

Construction:

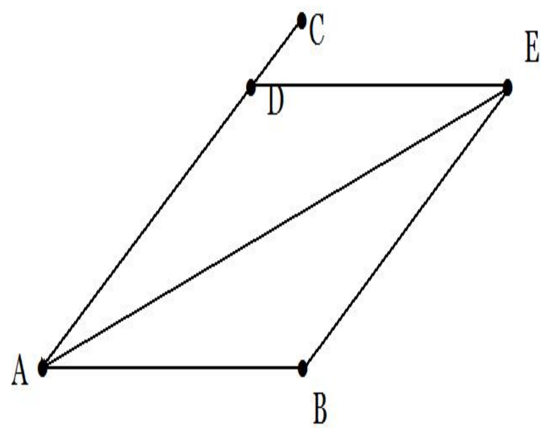
(1) Circle AB and get point D where the circle intersects the segment $\overline{AC}$



(2) Circle DA and BA to get point E where they intersect inside the angle. Use a straight edge to draw the segment $\overline{AE}$. We claim that $\overline{AE}$ bisects the angle BAC forming two equal angles BAE and EAC.



Proof. Using the rusty compass theorem and by drawing parallel lines, we can construct the rhombus $ABED$.

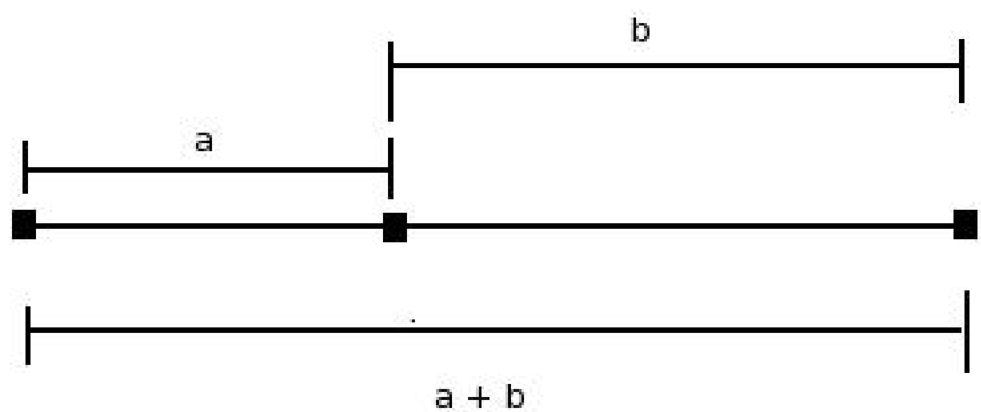


By properties of a rhombus, since $\overline{AE}$ is a diagonal of the rhombus $ABED$ it bisects the angle BAC.

□

2.2. **Constructing numbers from other constructible numbers.**

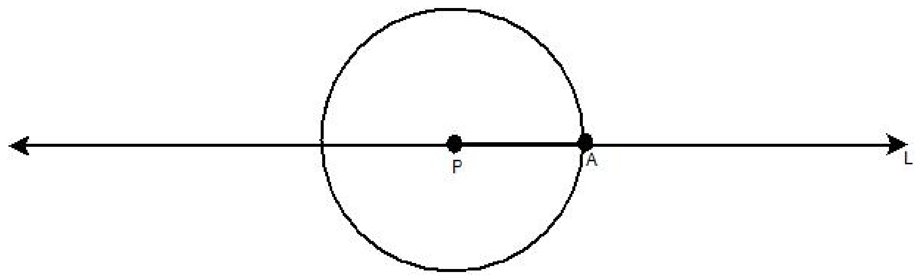
Theorem 2.7 (**Adding constructible numbers**). *If a and b are constructible numbers, $a + b$ is also a constructible number.*



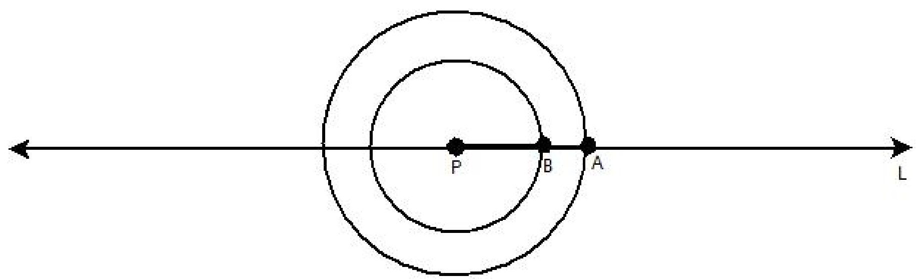
Theorem 2.8 (**Subtracting constructible numbers**). *Given constructible segments of lengths a and b. If $a \geq b$, then we can construct a segment of length $a - b$ using a straight edge and compass.*

Construction:

(1) Construct the line L using a straight edge. Choose a point P on L. Using a compass, construct a circle with center P and a radius of length a. Mark one of the two intersections of the circle with line L as A.



(2) Using a compass construct a circle with center P and a radius of length b.



(3) Mark the intersection of the circle and line L on the same side of point P as A; name B.

Proof. The length of $\overline{BA}$ is the length of a excluding the length of b. Therefore, $\overline{BA}$ has a length of $a - b$. □

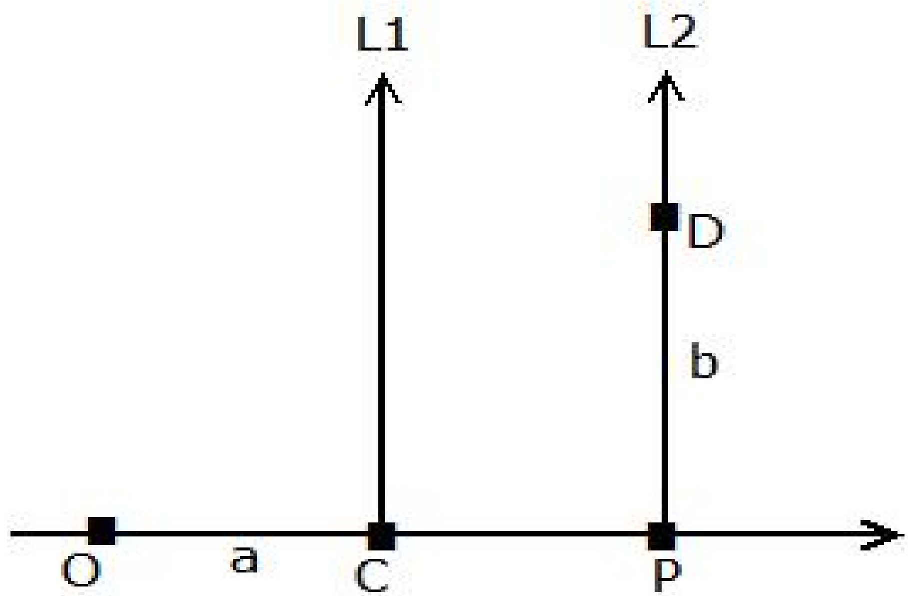
Theorem 2.9 (Multiplying constructible numbers). *Given segments of length a and b, we can construct a segment of length ab.*

Construction:

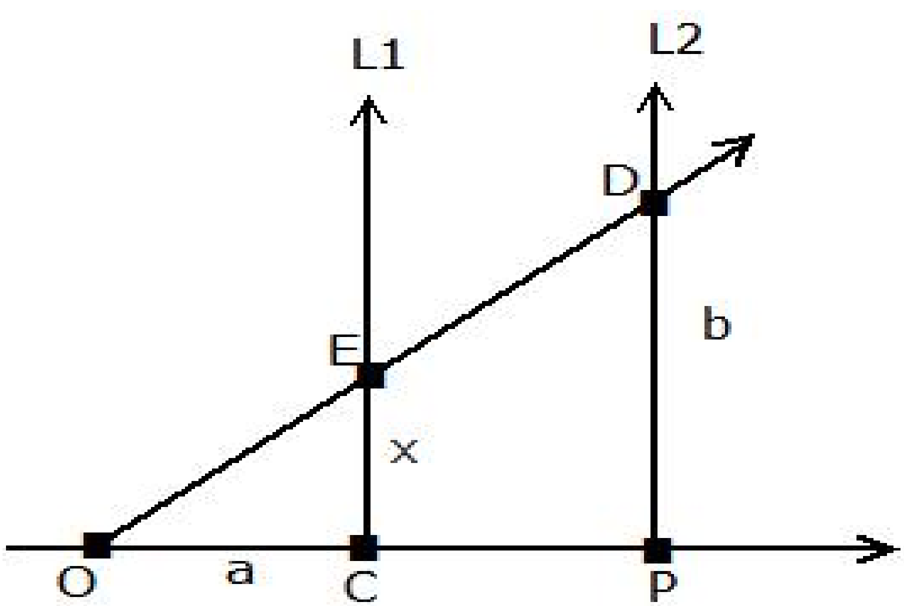
(1) Draw a line segment $\overline{OC}$ with length a and extend the line to point P so that the length of $\overline{OP}$ is 1. Thus if a and b are greater than 1, we must divide each by a single natural number larger than both a and b to get a and b less than 1.

(2) Raise two perpendicular lines at points C and P to get the lines $L1$ and $L2$

(3) Use the rusty compass theorem with length b to get D on $L2$ so that $\overline{PD}$ has length b.



(4) Draw the line segment from O to D producing point E at the intersection of $\overline{OD}$ and $L1$. The length of $\overline{EC}$ is x and we claim that x=ab



Proof. Because POD and COE are similar triangles, we can compare the ratio of the sides to each other giving the equation:

$$\frac{b}{x} = \frac{1}{a} \tag{2.1}$$

After multiplying both sided by x and a, we get:

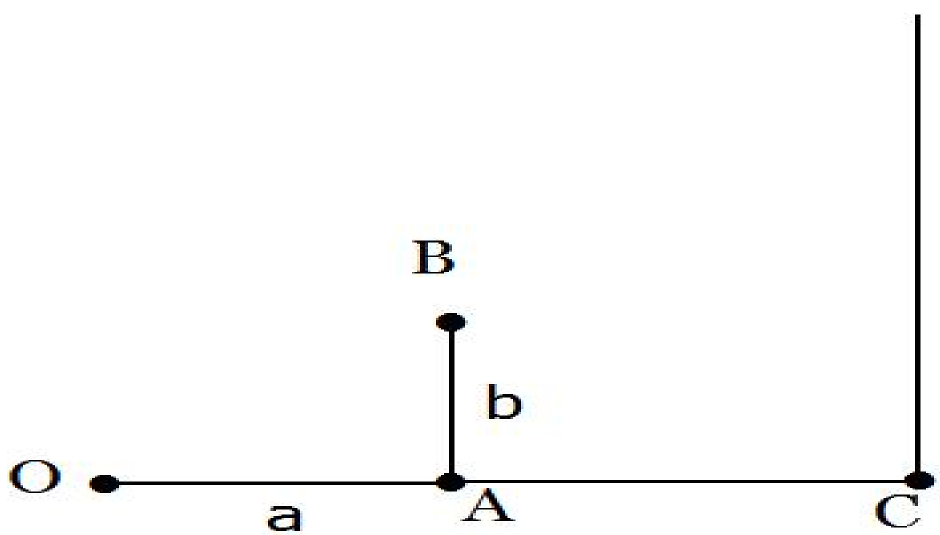
$$x = ab \tag{2.2}$$

□

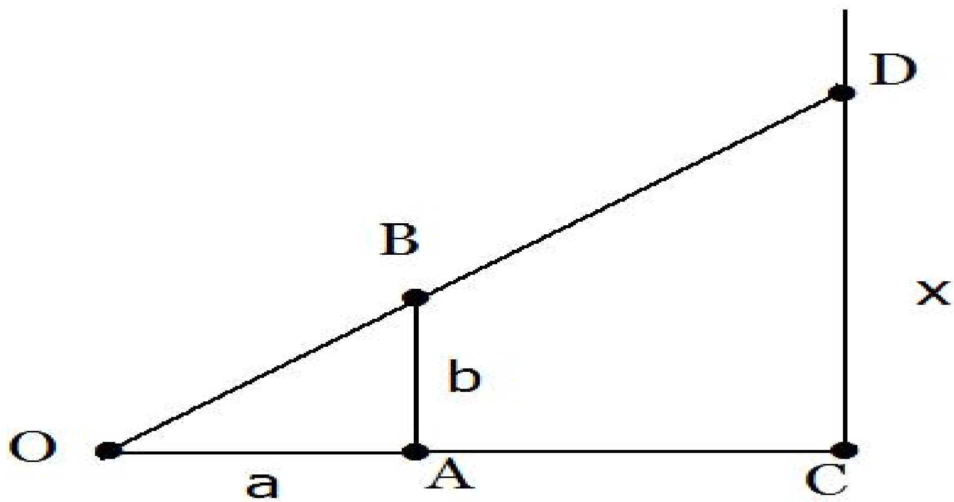
Theorem 2.10 (Dividing constructible numbers). *Given constructible segments of length a and b, we can construct a segment of length* $\frac{b}{a}$.

Construction:

(1) Draw a line segment $\overline{OA}$ with length a and extend the line to point C. The length of $\overline{OC}$ is 1. Thus if a and b are greater than 1, we must divide each by a single natural number larger than both a and b.



(2) At point A, raise a perpendicular of length b using the rusty compass theorem and get point B. At point C, raise another perpendicular.



(3) Use a straight edge to draw a line through points O and B and get D where the new line intersects with the line perpendicular to point C. We claim the segment $\overline{CD}$ has the length $\dfrac{b}{a}$.

Proof. When we look at the two similar triangles AOB and COD, we can find the length of x by setting the ratio of the sides adjacent to the angle COD equal to the ratio of the sides opposite of the angle and we get:

$$\frac{1}{a} = \frac{x}{b} \tag{2.3}$$

After multiplying both sides by b, we get:

$$x = \frac{b}{a} \tag{2.4}$$

Thus we have proved the claim and determined that $\dfrac{b}{a}$ is constructible given a,b are constructible numbers. □

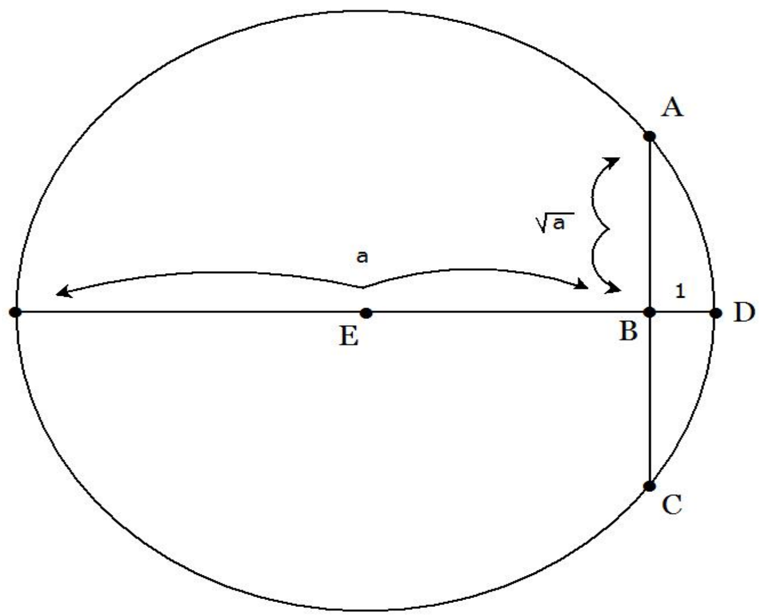
Theorem 2.11 (**Constructing the square root of a constructible number**). *Given a constructible number a, we can construct $\sqrt{a}$.*

Construction:

(1) Construct a circle of radius $\dfrac{a+1}{2}$.

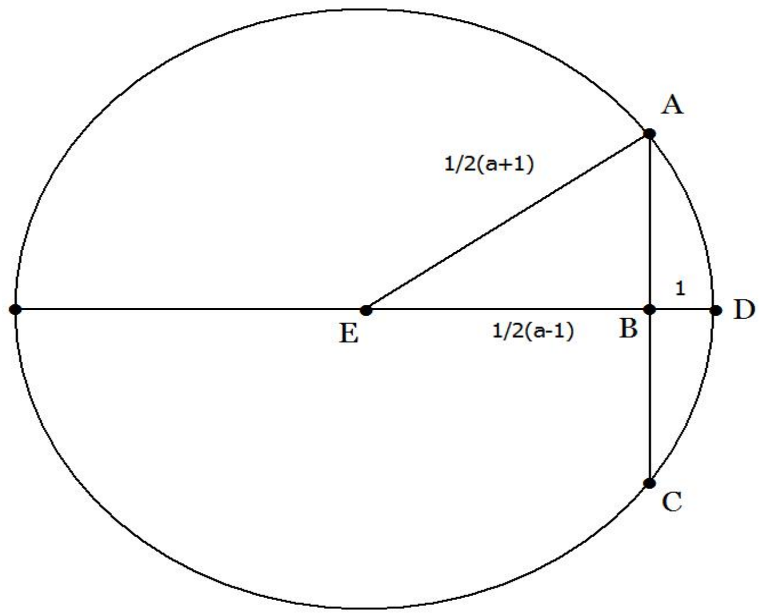
(2) Construct the diameter of the circle, which has the length $a+1$.

(3) Using a straight edge, construct a chord that is perpendicular to the diameter that splits the diameter into two segments of lengths a and 1.



We claim the length of the chord $\overline{AC}$ is $2\sqrt{a}$ and the length of both $\overline{AB}$ and $\overline{BC}$ is $\sqrt{a}$

Proof. We know the radius is $\dfrac{a+1}{2}$ so the length of $\overline{AE}$ is $\dfrac{a+1}{2}$. The length of $\overline{EB}$ is: $\dfrac{a+1}{2} - 1 = \dfrac{a-1}{2}$



By the Pythagorean theorem we have:

$$\sqrt{(\frac{a+1}{2})^2-(\frac{a-1}{2})^2}=\sqrt{\frac{a^2+2a+1-a^2+2a-1}{4}}=\sqrt{\frac{4a}{4}}=\sqrt{a}$$

Thus the length of $\overline{AB}$ is $\sqrt{a}$ □

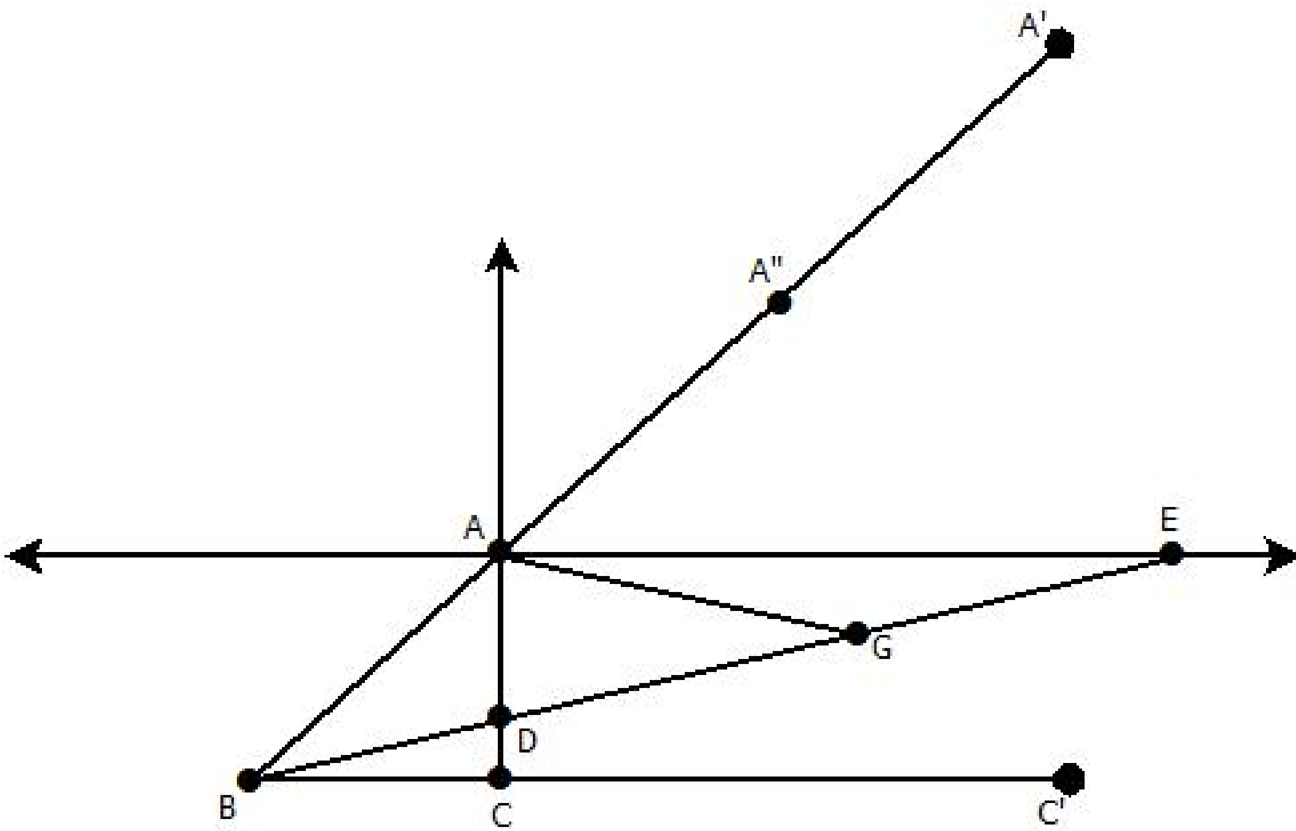
There are some constructions that have been proven impossible like squaring the cube and doubling the cube. Trisecting the angle is also impossible with just a straight edge and compass but is possible with the neusis construction.

3. Neusis Constructions

Definition 3.1. We are given that we can construct the origin (0,0) and the point (0,1) with a straightedge and compass. Since we can construct lines and circles using these two points, we can use our straightedge with two notches to slide and pivot along these lines and circles with our two notched straightedge to construct a point.

With a twice notched straight edge and compass, it is clear that all constructions using a regular straight edge and compass are still constructible.

Theorem 3.2 (**Trisecting the angle**). *Given an angle A'BC', we can trisect the angle.*



Construction:

(1) Using a straight edge with two marks, mark off the distance on $\overline{BA'}$ to get A''

(2) Bisect $\overline{BA''}$ get A

(3) Drop a perpendicular line from A to $\overline{BC'}$ to get C

(4) Draw a line parallel to $\overline{BC}$ through A to get L

(5) Pivot and slide on B so the marks on the straight edge are on $\overline{AC}$ and L to get D and E

(6) Line $\overline{BA}$, $\overline{DE}$, and $\overline{BA''}$ are equal

(7) Bisect $\overline{DE}$ and get G

(8) Draw $\overline{AG}$

(9) Claim Line $\overline{AG}$ and $\overline{DG}$ are equal

Proof. $\angle$ C'BA = $\angle$ DEA

$\angle$ DBA = $\angle$ DGA

$\angle$ DGA = 2$\angle$ GEA

If $\angle$ DBC= x

then $\angle$ ABD= 2x and $\angle$ A'BC' = 3x

□

Theorem 3.3. *(***Nicomedes***). Given a constructible length a, it is possible to find $\sqrt[3]{a}$ using a compass and twice-notched straightedge.*

Proof. First, assume that all notches are a unit, 1, distance apart from each other. Because we can construct inverses, it is enough to find $\sqrt[3]{a}$ for $a < 1$. We begin by constructing a rectangle with length 2 along the x-axis and width 2a along the y-axis, it is enough to find $\sqrt[3]{a}$. Construct the point E so that $\overline{EA}$ is perpendicular to $\overline{OA}$ and $|\overline{EB}| = 1$. Next, we construct the point F so that it a distance 2a away from O. Then, we construct the line l through B that is parallel to $\overline{EF}$. Using a straightedge with two notches, We place one notch on the line l, the other notch on on the line $\overline{OA}$, and we can adjust these points so the straight edge passes through E, giving us points G and H. Finally, we claim that the distance $x = |\overline{BH}|$ is $2\sqrt[3]{a}$. □

<u>Claim</u> We can calculate $x = 2\sqrt[3]{a}$ by using similar triangles and proportions.
By using the Pythagorean Theorem, we obtain the values of $\triangle BAE$ and $\triangle HAE$.

$$\begin{array}{cc} \triangle BAE & |AE|^2 + a^2 = 1 \\ \triangle HAE & |AE|^2 + (x+a)^2 = (y+1)^2 \end{array}$$

Using elimination, we subtract the two pythagorean triangle equations creating the following equation.

$$a^2 - (x-a)^2 = 1 - (y+1)^2$$

Then, we use further calculations to obtain a proportion.

$$\begin{gathered} a^2 - (x^2 + 2ax + a^2) = 1 - (y^2 + 2y + 1) \\ (-x)^2 - 2ax = (-y)^2 - 2y \\ x(x+2a) = y(y+2) \end{gathered}$$

$$\frac{x}{y} = \frac{y+2}{x+2a}$$

Since $L \parallel \overline{EF}$ *in* $\triangle HEF$*. Then,* $\triangle HEF$ *is similar to* $\triangle HEB$

$$\frac{y}{1} = \frac{4a}{x}$$

Since $\triangle HBC$ *is similar to* $\triangle CDJ$

$$\frac{x}{2a} = \frac{2}{|DJ|} \therefore |DJ| = \frac{4a}{x}$$

Since $\triangle HOJ$ *is similar* $\triangle CDJ$

$$\frac{x+2a}{2a} = \frac{y+2}{y} \therefore \frac{y+2}{x+2a} = \frac{y}{2a}$$

Therefore, we can calculate x

$$\frac{x}{y} = \frac{y+2}{x+2a} = \frac{y}{2a}$$

$$\frac{x}{y} = \frac{y}{2a} \therefore y = \frac{4a}{x}$$

Then, we plug y *back into the equation of* $\frac{x}{y} = \frac{y}{2a}$ *which gives us:*

$$\frac{x}{\frac{4a}{x}} = \frac{\frac{4a}{x}}{2a} \therefore \frac{x^2}{4a} = \frac{2}{x}$$

After simplifying, we find:

$$x^3 = 8a$$
$$x = 2\sqrt[3]{a}$$

4. Fields and constructible numbers

Definition 4.1. A number w is constructible if we can use a straight edge and compass to construct a segment of length w.

Definition 4.2. A number w is neusis constructible if we can use a straight edge with two notches and compass to construct a segment of length w.

We can talk about the set of both constructible numbers and neusis constructible numbers and form more general conclusions about what real numbers are constructible by talking about **fields** and field extensions.

A **field** is a set with two binary operations: addition and multiplication that satisfy the following:

- The addition, subtraction, multiplication or division of any two elements in the field is also in the field.
- Addition and multiplication are both associative:

$$a + (b + c) = (a + b) = c$$
$$a(bc) = (ab)c$$

- Addition and multiplication are both commutative:

$$a + b = b + a$$
$$ab = ba$$

- There exist identities
  (1) $\exists 0$ such that $a + 0 = a \ \ \forall a$ which is known as the additive identity
  (2) $\exists 1$ such that $a * 1 = a \ \ \forall a$ which is known as the multiplicative identity
- There exist inverses
  (1) $\forall a \ \ \exists(-a)$ such that $a + (-a) = 0$
  (2) $\forall a \neq 0 \ \ \exists a^{-1}$ such that $a(a^{-1}) = 1$
- Addition and multiplication are distributive:

$$\forall a, b, c \ \ a(b + c) = ab + ac$$

- Examples of Fields
  - $\mathbb{Q}$, the rational numbers
  - $\mathbb{R}$, the real numbers
  - $\mathbb{C}$, the complex numbers

Theorem 4.3. *The set of all numbers constructible with a straight edge and compass is a field.*

Proof.
- 0,1 are both constructible numbers.

- The addition, subtraction, multiplication and division of two constructible numbers is also constructible.
- Addition and multiplication are both associative and commutative within the set.
- There exists an additive and multiplicative inverse for each constructible number.

□

Theorem 4.4. *The set of all constructible numbers with a twice notched straight edge and compass is a field.*

Proof.
- We can construct all constructible numbers with with a straight edge and compass with the twice notched straight edge and compass as well as extra constructible numbers.
- 0,1 are both constructible numbers with a twice notched straight edge and compass.
- The addition, subtraction, multiplication and division of two constructible numbers is also constructible.
- Addition and multiplication are both associative and commutative within the set.
- There exists an additive and multiplicative inverse for each constructible number with a twice notched straight edge and compass.
- We conclude that the field of all constructible numbers with a straight edge and compass is a subfield of the field of all constructible numbers with a twice notched straight edge and compass.

□

Theorem 4.5. *The number $\alpha \in \mathbb{R}$ is constructible with straight edge and compass ($\alpha \in constructiblenumbers$) if and only there is a sequence of field extensions*

$$\mathbb{Q} = F_0 \subset F_1 \subset F_2 \subset \ldots\ldots \subset F_n \textit{ so that } [F_i : F_{i-1}] = 2 \textit{ or } 1 \textit{ for}$$

$$i = 1, \ldots\ldots., n \textit{ (i.e. } F_i = F_{i-1}(\sqrt{\beta_i})\textit{)}, \beta_i \in F_{i-1} \textit{ and } \alpha \in F_n$$

Examples :

$(\mathbb{Q})$ is constructible with only a straight edge and compass.

$\sqrt{\dfrac{p}{q}}$ is constructible with only a straight edge and compass

$\sqrt{\sqrt{\dfrac{p}{q}}} = \sqrt[4]{\dfrac{p}{q}}$ is constructible with only a straight edge and compass.

$\sqrt[2n]{\dfrac{p}{q}}$ is constructible with only a straight edge and compass.

$\sqrt{\sqrt{3} + \sqrt{5} + 2 + \sqrt[4]{7}}$ is constructible with only a straight edge and compass.

<u>Analogous Statement</u>

Theorem 4.6. *If a number $\alpha \in (\mathbb{R})$ is in F_n so that there is a sequence of field extensions*

$$(\mathbb{Q}) = F_0 \subset F_1 \subset \ldots\ldots \subset F_n \text{ with } [F_i : F_{i-1}] = 1, 2, 3$$

for $i = 1, \ldots\ldots, n$, then α is constructible with straight edge with two notches and compass (i.e. $\alpha \in$ the constructible numbers with two notches.)

Note : The converse is false. We could potentially create $\sqrt[5]{a}$ with the neusis construction which cannot be written as a sequence of field extensions with degree 1, 2 or 3.

Example : We can construct the number 2.515709455 with the neusis construction because we can write the number as $\dfrac{5 - \sqrt[3]{3}}{\sqrt{2}}$

5. Conclusion

We have shown the basic operations for constructing numbers with a compass and straight edge as well as more complicated constructions. Some constructions like trisecting an angle or constructing the cube root of a number are impossible with only a straight edge and compass but can be done using the neusis construction. Both the set of constructible numbers and the set of numbers constructible using the neusis construction are fields. This allows us to create a tower of field extensions which allows us to construct most real numbers as long as the degree of each extension is 1, 2 or 3. Using field extensions, the degree for the whole tower is $2^q 3^p$. However we do not know for certain if we can construct a number not in this form such as $\sqrt[5]{a}$ where a is a constructible number.

References

[1] Baragar, Arthur, 2002: Constructions Using a Compass and Twice-Notched Straightedge. *The American Mathematical Monthly*, **1**09, 151-164.

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