LaPlace Transforms Project 2

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The Gamma Function - Research Forthcoming

The Laguerre Polynomial

el.jpg

The Laguerre equation is given by:

$$ty'' + (1 - t)y' + ny = 0$$

where $y(0) = 1$ and n is non-negative integer

The Laguerre Polynomial

1. The Laguerre Polynomial of order n

$$l_n(t) = \sum_{k=0}^{n} \frac{n!(-1)^k t^k}{(k!)^2(n-k)!}$$

2. For n $\neq$ m,

$$\int_0^\infty e^{-t} l_n(t) l_m(t) dt = 0$$

The Laguerre Polynomials

The first few polynomials are:

1. $l_0 = 1$
2. $l_1 = -t + 1$
3. $l_2 = \frac{1}{2}(t^2 - 4t + 2)$
4. $l_3 = \frac{1}{6}(-t^3 + 9t^2 - 18t + 6)$

The Laguerre Polynomials on a Graph

C:/Latek/Laguerre_Graph.jpg

Differential Operators and Relationships

1. $E_{-} = tD^2 + D$
2. $E_{-} l_n = -n l_{n-1}$
3. $E_O = 2tD^2 + (2-2t)D - 1$
4. $E_O l_n = -(2n+1) l_n$
5. $E_{+} = tD^2 + (1-2t)D + (t-1)$
6. $E_{+} l_n = -(n+1) l_{n+1}$
7. LIE BRACKET $[E_0, E_+] = E_0 E_+ - E_+ E_0 = -2E_0$

The Laplace Transform

1. $L_n = \mathcal{L}\{l_n(t)\} = \frac{(s-1)^n}{s^{n+1}}$
2. $\mathcal{L}\{l_n(at)\} = \frac{(s-a)^n}{s^{n+1}}$

Laguerre Polynomial Properties

1. $\int_0^t l_n(x)dx = l_n(t) - l_{n+1}(t)$
2. $\int_0^t l_n(x)l_m(t-x)dx = l_{m+n}(t) - l_{m+n+1}(t)$
3. $l_{n+1}(t) = \frac{1}{n+1}[(2n+1-t)l_n(t) - nl_{n-1}(t)]$

Solving Laplace Transforms Using DE's

The Laplace Transform of a function, $F(t)$, *is*

$$f(s) = \int_0^\infty e^{-at} F(t) dt.$$

Therefore,

Theorem

If $f(s) = L\{F(t)\}$, *then*

1. $sf(s) - F(0) = L\{F'(t)\}$
2. $f'(s) = L\{-tF(t)\}$

Theorems

Theorem

If $X(t) \sim At^{\alpha}(\alpha > -1)$ *as* $t \to 0$, *then*

$$x(s) \sim \frac{A\Gamma(\alpha + 1)}{s^{\alpha+1}} \quad as \quad s \to \infty$$

Theorem

If $X(t) \sim Bt^{\beta}(\beta > -1)$ *as* $t \to \infty$,

$$x(s) \sim \frac{B\Gamma(\beta + 1)}{s^{\beta+1}} \quad as \quad s \to 0$$

Table of Laplace Transforms

$f(s)$	$F(t)$
x	X
$sx - X(0)$	X'
$s^2x - sX(0) - X'(0)$	X''
$-x'$	tX
$-sx' - x$	tX'
$-s^2x' - 2sx + X(0)$	tX''
x''	t^2X
$sx'' + 2x'$	t^2X'
$s^2x'' + 4sx' + 2x$	t^2X''

Table: Basic Laplace Transforms

Laplace Transform Example

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Theorem

The Laplace Transform of the zeroth-order Bessel Function is the following:

$$L\{J_0(t)\} = \frac{1}{\sqrt{s^2+1}}$$

and the Laplace Transform of the first-order Bessel Function is:

$$L\{J_1(t)\} = \frac{\sqrt{s^2+1}-s}{\sqrt{s^2+1}}$$

Proving the Theorem

Proof of Theorem.

Given Equation	$X(t) = J_0(a\sqrt{t})$
Differential Equation	$4tX'' + 4X' + a^2X = 0$
Laplace Transform	$x' + \frac{4s-a^2}{4s^2}x = 0$
Integrating Factor	$I = e^{\frac{a^2}{4s}}s$
Solve Diff. Equation	$x(s) = \frac{ce^{\frac{-a^2}{4s}}}{s}$
Use Theorem 2 to find c.	$x(s) \sim 1/s \; as \; s \to \infty$
Plug in $c = 1$	$L\{J_0(a\sqrt{t})\} = \frac{e^{\frac{-a^2}{4s}}}{s}.$

□

Table of Inverse Laplace Transforms

$f(s)$	$F(t)$
x	X
x'	$-tX$
x''	t^2X
$sx - X(0)$	X'
sx'	$-tX' - X$
sx''	$t^2X' + 2tX$
$s^2x - sX(0) - X'(0)$	X''
$s^2x' + X(0)$	$-tX'' - 2X'$
s^2x''	$t^2X'' + 4tX' + 2X$

Table: Inverse Laplace Transforms

Inverse Laplace Transform Example

Theorem

The Bessel Differential Equation of the n^{th} order can be defined as

$$t^2X'' + tX' + (t^2 - n^2)X = 0$$

where the solutions of this differential equation are the n^{th} order Bessel Functions.

Proving the Theorem

Proof of Theorem.

Given Equation	$x(s) = \frac{(\sqrt{s^2+1}-s)^n}{\sqrt{s^2+1}}$
Diff. Equation	$(s^2+1)x'' + 3sx' + (1-n^2)x = 0$
Inverse Laplace	$t^2X'' + tX' + (t^2 - n^2)X = 0$
Bessel's Equation	$X(t) = J_n(t)$

□

Solving Difference Equations

$$a_{n+2} + ba_{n+1} + ca_n = f(n)$$

$$y(t) = a_n, \quad f(t) = f(n), for \quad n \leq t < n+1$$

$$y(t+2) + by(t+1) + cy(t) = f(t)$$

$$\mathcal{L}\{y(t+1)\} = e^s Y(s) - \frac{a_0 e^s (1 - e^{-s})}{s}$$

$$\mathcal{L}\{y(t+2)\} = e^{2s} Y(s) - \frac{e^s (1 - e^{-s})(a_0 e^s + a_1)}{s}$$

$$\mathcal{L}\{f(t)\} = \frac{1 - e^{-s}}{s} \cdot \sum_{k=0}^{\infty} e^{-sk} f(k)$$

Solving Difference Equations - 2

Theorem

1. $$a_n = A\alpha^n + B\beta^n$$

2. $$a_n = (An + B)\alpha^n$$

3. $$r_1 = \alpha + i\beta, \quad r_2 = \alpha - i\beta$$
$$a_n = u^n(A\cos(n\theta) + B\sin(n\theta))$$

with $u = \sqrt{\alpha^2 + \beta^2}$, $\theta = \sin^{-1}\frac{\beta}{u}$

Solving Difference Equations - 3

$$a_{n+2} - a_{n+1} - a_n = 0, \quad a_0 = 0, \quad a_1 = 1$$

$$Y(s) = \frac{e^s(1-e^{-s})}{s} \cdot \frac{1}{(e^s-\alpha)(e^s-\beta)}$$

$$\begin{aligned}
Y(s) &= \frac{e^s(1-e^{-s})}{s(\alpha-\beta)} \cdot \left(\frac{1}{e^s-\alpha} - \frac{1}{e^s-\beta}\right) \\
&= \frac{1-e^{-s}}{s(\alpha-\beta)} \cdot \left(\frac{1}{1-\alpha e^{-s}} - \frac{1}{1-\beta e^{-s}}\right) \\
a_n &= \frac{1}{\alpha-\beta} \cdot (\alpha^n - \beta^n) \\
a_n &= \frac{1}{\sqrt{5}} \cdot \left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]
\end{aligned}$$

Solving Difference Equations - 4

Theorem

Let $a_{n,p}$ be a fixed particular solution to the second order linear recursion relation

$$a_{n+2} + ba_{n+1} + ca_n = f(n).$$

Then any other solution has the form $a_n = a_{n,h} + a_{n,p}$, for some homogeneous solution $a_{n,h}$.

Solving Difference Equations - 5

$$a_{n+2} + ba_{n+1} + ca_n = f(n)$$

$$(e^{2s} + be^s + c)Y(s) = \frac{p(e^{-s})}{sq(e^{-s})}$$

$$Y(s) = \frac{1}{s} \cdot \frac{p(e^{-s})}{\prod(1 - r_k e^{-s})^p e^{-sq}}$$

$$Y(s) = \frac{1}{s} \cdot \sum \frac{c_i}{(1 - r_k e^{-s})^p}$$

$$\sum_{k=0}^{\infty} e^{-sk} H(k,p) = \frac{1}{(1 - re^{-s})^p}$$

$$\frac{r^n}{(p-1)!} \cdot \prod_{k=1}^{p-1}(n+k) = H(n,p) = \mathcal{L}^{-1}\{\frac{1 - e^{-s}}{s(1 - re^{-s})^p}\}$$

Solving Difference Equations - 6

$$a_{n+2} - 5a_{n+1} + 6a_n = n2^n, \quad a_0 = 0, \quad a_1 = 1$$

$$\begin{aligned} \mathcal{L}\{f(t)\} &= \frac{1-e^{-s}}{s} \cdot \sum_{k=0}^{\infty} e^{-sk} k2^k \\ &= \frac{1-e^{-s}}{s} \cdot \left(\sum_{k=0}^{\infty} e^{-sk}(k+1)2^k - \sum_{k=0}^{\infty} e^{-sk} 2^k \right) \\ &= \frac{1-e^{-s}}{s} \cdot \left(\sum_{k=0}^{\infty} e^{-sk} H(k,2) - \sum_{k=0}^{\infty} (2e^{-s})^k \right) \end{aligned}$$

$$\mathcal{L}\{f(t)\} = \frac{1-e^{-s}}{s} \cdot \left(\frac{1}{(1-2e^{-s})^2} - \frac{1}{1-2e^{-s}} \right)$$

Solving Difference Equations - 7

$$
\begin{aligned}
Y(s)(e^{2s} - 5e^{s} + 6) &= \frac{1 - e^{-s}}{s} \cdot \left(\frac{1}{(1 - 2e^{-s})^2} \right. \\
&\quad \left. - \frac{1}{1 - 2e^{-s}} + e^{s} \right) \\
Y(s) &= \frac{1 - e^{-s}}{s} \cdot \left(\frac{3}{1 - 3e^{-s}} - \frac{5}{2(1 - 2e^{-s})} \right. \\
&\quad \left. - \frac{1}{2(1 - 2e^{-s})^3} \right) \\
a_n &= 3^{n+1} - \frac{5}{2} 2^{n} - \frac{1}{2} H(n,3)\{r = 2\} \\
a_n &= 3^{n+1} - (n^2 + 3n + 12).2^{n-2}
\end{aligned}
$$

Solving Difference Equations - 8

Theorem

$$a_{n+2} + ba_{n+1} + ca_n = g(n)r^n, \quad deg(g) = k$$
$$x^2 + bx + c = 0$$

1. $$a_{n,p} = g^*(n)r^n, \quad deg(g^*) = k$$
2. $$a_{n,p} = g^*(n)r^n, \quad deg(g^*) = k + 1$$
3. $$a_{n,p} = g^*(n)r^n, \quad deg(g^*) = k + 2$$

Solving Difference Equations - 9

$$a_{n+2} + ba_{n+1} + ca_n = c_1 \sin(kn) + c_2 \cos(kn)$$

$$a_{n,p} = c_3 \sin(kn) + c_4 \cos(kn)$$

Solving Difference Equations - Conclusion

$$a_{n+2} - 4a_{n+1} + 4a_n = (n^2 + 1)2^n + n3^n + 2\sin(4n) + 3\cos(n)$$

$$a_n = g(n)2^n + h(n)3^n + c_1 \sin(4n) + c_2 \cos(4n) + c_3 \sin(n) + c_4 \cos(n)$$

with $deg(g) = 4$ and $deg(h) = 1$.

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