REMARKS ON THE ZETA FUNCTION OF A GRAPH

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Abstract. We make two observations about the zeta function of a graph. First we show how Bass's proof of Ihara's formula fits into the framework of torsion of complexes. Second, we show how in the special case of those graphs that are quotients of the Bruhat-Tits tree for SL(2, K) for a local nonarchimedean field K, the zeta function has a natural expression in terms of the L-functions of Coexter systems.

1. Introduction

At the Conference itself I gave an overview of the origins of the concept of the zeta function of a graph in the works of Ihara, and its motivation in trying to understand the zeta functions of modular curves. Let

$$\Gamma \subset \mathrm{SL}(2, \mathbf{Q}_p)$$

be a discrete cocompact and torsion-free subgroup. Let X be the Bruhat-Tits building associated to SL(2, $\mathbf{Q}_p$). Ihara's zeta function can be interpreted as the zeta function of the finite quotient graph $X_\Gamma = \Gamma\backslash X$. In certain cases, Ihara discovered an essential identity between the zeta function of the graph X_Γ and the zeta function of a certain Shimura curve $\mathcal{X}_\Gamma$ reduced modulo the prime p. For precise statements of Ihara's results, see [17], [18], [19]. The relation between the graph and the modular curve is via the theory of automorphic forms. There are two salient points: (1) the zeta function of a modular curve (say for a congruence subgroup $\Gamma_0(N)$) can be computed via the action of the Hecke operator T_p on the space of cusp forms of weight 2, and (2) the cusp forms come from a quaternion algebra over $\mathbf{Q}$ by the Jacquet-Langlands correspondence, and the Hecke operator can be interpreted as the adjacency operator of a finite graph that is a quotient of the Bruhat-Tits tree for SL(2, $\mathbf{Q}_p$) (see [21, Ch. 9]). The group Γ comes from the p-units in an order for the quaternion algebra.

Ihara's results became one of the origins of the studies of the zeta functions of Shimura varieties by Langlands, Rapoport and Kottwitz, which has culminated in an "explicit" formula for the latter, at least for those Shimura varieties that are moduli spaces of abelian varieties with additional structures (PEL types). A good exposition of these results can be found in [23]. These formulas are very complicated and hard to compute in examples, so it is desirable to have a more elementary and combinatorial description.

The aim is to generalize this connection from Shimura curves to Shimura varieties attached to higher-rank reductive algebraic groups G over $\mathbf{Q}$. The idea is that the Euler p-factor of the zeta function of the Shimura variety attached to G (assuming G is of the sort that it has a Shimura variety) should be related to a combinatorially

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defined zeta function associated to the Bruhat-Tits building of the group $G(\mathbf{Q}_p)$. However, even before this can be attempted, a reasonable definition of the zeta function for the finite quotients of the building X of $G(\mathbf{Q}_p)$ must be given. There are several ideas in this direction, two of which were discussed at the Conference by Cristina Ballantine and Winnie Li. There is another approach, based on the Selberg trace formula, by Anton Deitmar (see [7], [8], [9]). The relations among these approaches and to the theory of Shimura varieties remains unclear. Over and above the connection to Shimura curves, zeta functions of graphs are interesting in their own right, and the search for higher-dimensional analogs seems worthwhile.

In this note I want to make two observations that do not seem to have been noticed before concerning zeta functions of graphs. Before doing so, let us recall the definitions. Let Y be a finite graph. If Y is connected, the universal covering space $\tilde{Y}$ is a tree on which the group $\Gamma = \pi_1(Y)$ acts fixed-point free and with quotient Y. Let $\rho : \Gamma \to \mathbf{GL}(V)$ be a finite-dimensional representation, then the zeta (or L) function is

$$Z(Y, \rho, u) = \prod_{\gamma \in \mathfrak{P}(\Gamma)} \frac{1}{\det(1 - \rho(\gamma) u^{\deg(\gamma)})}$$
$$= \exp\left(\sum_{m=1}^{\infty} N_m(Y, \rho) u^m / m \right).$$

where the product ranges over all the equivalence classes of primitive cycles in the graph Y. Recall that if Y is connected, there is a canonical bijective correspondence of $\mathfrak{P}(\Gamma)$ with the primitive conjugacy classes in $\pi_1(Y)$. Here,

$$N_m(Y, \rho) = \sum_{\deg(\gamma) | m} \deg(\gamma) \mathrm{Tr}(\rho(\gamma^{m/\deg(\gamma)})).$$

When $\rho = 1$ is the trivial representation, this is the number of closed, backtrackless (or reduced) and tail-less paths in Y of length m (see [24], [25]). Bass's treatment [2] is more general in that he allows an action of a group Γ on a tree $\tilde{Y}$ which has fixed points with finite isotropy groups, but then the interpretation of the zeta function in terms of the quotient graph is more complicated. In the discussion that follows, we will restrict to the case where $\rho = 1$ and the isotropy groups are trivial, as this allows a more elementary exposition.

In general we will follow the notations of Bass's paper [2]. Let VX be the set of r_0 vertices of the graph X, and EX be the set of oriented edges. If r_1 is the number of geometric edges of X, the cardinality of EX is $2r_1$. There is an orientation reversal $J : EX \to EX$, and boundary maps $\partial_0, \partial_1 : EX \to VX$. Recall that a path in X of length n is a sequence $c = (e_1, \ldots, e_n)$ of edges such that $\partial_0 e_i = \partial_1 e_{i-1}$ for all $0 < i \leq n$, and that the path is reduced if $e_i \neq J(e_{i-1})$ for all $0 < i \leq n$. We write $(e_1, \ldots, e_n)red$ to indicate that the path is reduced. The path is closed if $\partial_1 e_n = \partial_0 e_1$, and a closed path has no tail if $e_n \neq J(e_1)$. Two paths are equivalent if one is gotten from the other by a cyclic shift of the edges.

Let $C_0(X)$ (resp. $C_1(X)$) be the $\mathbf{C}$-vector space spanned by VX (resp. EX). There are a number of operators on these spaces and identities connecting them (see [2, pp. 744-751]). We recall the endomorphism T of $C_1(X)$ defined by

$$T(e) = \sum_{(e, e_1) red} e_1$$

It is easily shown that

$$Z(X,u) = \frac{1}{\det(1-uT)}, \tag{1}$$

which is equivalent to the identity

$$\mathrm{Tr}(T^m) = \#\{\text{closed backtrackless tail-less paths of length } m\} = N_m(Y).$$

This is Bass's version of a formula discovered by Hashimoto, [13], [14], [15], [16], and see the proof of proposition 3.3.

For each $x \in VX$ we let

$$d(x) = \text{degree of } x = \#\{e \in EX \mid \partial_0 e = x\},$$

$q(x) = d(x) - 1$ and define an operator $Q : C_0(X) \to C_0(X)$ via $Q(x) = q(x)x$, which is the diagonal matrix whose terms are the vertex degrees minus 1. The adjacency matrix is defined as

$$\delta(x) = \sum_{\substack{e \in EX \\ \partial_0 e = x}} \partial_1 e$$

Then Bass's generalization of Ihara's formula for the zeta function of a graph is

$$Z(X,\, u) = \frac{(1-u^2)^{\chi}}{\det(1-\delta u + Qu^2)} \tag{2}$$

where $\chi = r_0 - r_1$ is the Euler characteristic of the graph (see also [26]).

2. Bass's proof of Ihara's formula

Comparison of equations (1) and (2) shows

$$\frac{\det(1-\delta u + Qu^2)}{\det(1-uT)} = (1-u^2)^{\chi} \tag{3}$$

This is very reminiscent of an index formula. There are proofs of Ihara's formula via Selberg trace formula methods ([1], [27, Ch. 24], [28], [29]). In some cases, the Selberg trace formula can be related to geometric trace formulas in Lefschetz style, this being a key point in the study of Shimura varieties and moduli spaces of shtuka. I am unable to give a conceptual geometric proof of the identity (3), but I can show how Bass's proof of Ihara's formula, which seems like an unmotivated manipulation of identities, can be reformulated in terms of the torsion of complexes, an idea that is central in many proofs of index formulas. Actually, all the formulas necessary for this can be found in Bass's paper, but Bass himself makes no comment on any of this.

Proposition 2.1. *Let*

$$C_* = \Big[0 \longrightarrow C_n \xrightarrow{d_n} C_{n-1} \longrightarrow \ldots C_1 \xrightarrow{d_1} C_0 \xrightarrow{d_0} 0\Big]$$

be a chain complex of finite-dimensional vector spaces over a field K. Suppose that

$$\Phi = \{\Phi_i\} : C_* \longrightarrow C_*$$

is a chain endomorphism. This induces an endomorphism on the homology of the complex, and we have an identity

$$\prod_{i=0}^{n} \det(\Phi \mid C_i)^{(-1)^i} = \prod_{i=0}^{n} \det(\Phi \mid H_i(C_*))^{(-1)^i}$$

provided it makes sense, ie., none of the determinants is 0.

This is no doubt well-known; the additive form of it, namely the identity

$$\sum_i (-1)^i \mathrm{Tr}\,(\Phi \mid C_i) = \sum_i (-1)^i \mathrm{Tr}\,(\Phi \mid H_i(C_*))$$

is a key step in many proofs of fixed-point/index formulas. The proof of these identities is easily done by breaking the chain complex into short exact sequences, stable for Φ,

$$0 \longrightarrow \mathrm{Ker}(d_i) \longrightarrow C_i \longrightarrow \mathrm{Im}(d_i) \longrightarrow 0$$

and

$$0 \longrightarrow \mathrm{Im}(d_{i+1}) \longrightarrow \mathrm{Ker}(d_i) \longrightarrow H_i(C_*) \longrightarrow 0$$

then concatenating.

Corollary 2.2. *Let K be the field of real (or complex) numbers, and assume that each C_i has a nondegenerate symmetric (or Hermitian) inner product. Let δ_i be the adjoints of the operators d_i and*

$$\Delta_i = \delta_i d_i + d_{i+1}\delta_{i+1}$$

be the corresponding Laplacians. Then

$$\prod_{i=0}^{n} \det(t - \Delta_i \mid C_i)^{(-1)^i} = t^{\chi}$$

where $\chi = \sum_i (-1)^i \dim C_i = \sum_i (-1)^i \dim H_i(C_*)$ is the Euler characteristic of the complex.

Proof. By the Hodge theorem, $H_i(C_*)$ is isomorphic to the space of Δ_i-harmonic forms, ie., to those elements of C_i annihilated by Δ_i. Define $\Phi_i = t - \Delta_i$. This is a chain endomorphism because

$$\Delta_i d_{i+1} = d_{i+1}\delta_{i+1} d_{i+1} = d_{i+1}\Delta_{i+1}$$

The corollary now follows from the proposition, because the Laplacians acting trivially on homology, $\det(\Phi \mid H_i(C)) = t^{\dim H_i(C)}$. □

Let $C_*(X)$ be the complex associated to a graph X as above, with differential $\partial_0 - \partial_1$. The corollary says in this case that spectrum of Δ_1 provides no information not already contained in the spectrum of Δ_0. Recall that for the trivial metric for which the vertices (resp. edges) form an orthonormal basis, $\Delta_0 = 1 + Q - \delta$. Define the operator $\partial(u) = u\partial_0 - \partial_1$, where u is a formal parameter. This is a deformation of the differential $\partial(1)$ in the original complex. Let $\sigma_0 : C_0(X) \to C_1(X)$ be defined as

$$\sigma_0(x) = \sum_{\substack{e \in EX \\ \partial_0 e = x}} e$$

and a perturbation of this by $\sigma(u) = u\sigma_0$. Let $\Phi_0 = \Delta(u) = 1 - \delta u + Qu^2$, $\Phi_1 = (1 - Tu)(1 - Ju)$.

Lemma 2.3. 1. *Φ is a chain endomorphism of the complex $C_*(X), \partial(u)$.*
2. *$\sigma(u)$ is a chain-homotopy between Φ and the scalar endomorphism given by $(1-u^2)$, ie.,*

$$\partial(u)\sigma(u) = \Phi_0 - (1-u^2)$$
$$\sigma(u)\partial(u) = \Phi_1 - (1-u^2)$$

Proof. 1. According to the formulas on pp. 747-748 of [2],

$$\partial(u)\Phi_1 = \partial(u)\sigma(u)\partial(u) + (1-u^2)\partial(u) = \Phi_0\partial(u).$$

2. This is the formula (15) on p.747 and the formula just preceding section 1.4 on p. 748 of *loc. cit.* □

Applying proposition 2.1 to this situation, and using the fact that chain-homotopic maps induce the same map in homology, we see:

$$\begin{aligned}\frac{\det(1-\delta u+Qu^2)}{\det(1-Tu)\det(1-Ju)} &= \frac{\det(\Phi \mid C_0(X))}{\det(\Phi \mid C_1(X))}\\ &= \frac{\det(\Phi \mid H_0(C_*(X))}{\det(\Phi \mid H_1(C_*(X))}\\ &= \frac{\det(1-u^2 \mid H_0(C_*(X))}{\det(1-u^2 \mid H_1(C_*(X))}\\ &= \frac{\det(1-u^2 \mid C_0(X))}{\det(1-u^2 \mid C_1(X))}\\ &= (1-u^2)^{r_0-2r_1}\end{aligned}$$

Bass calculated by elementary means the formula $\det(1-Ju) = (1-u^2)^{r_1}$, so this completes the proof of Ihara's formula.

3. L-functions of Coxeter systems

Let (W, S) be a Coxeter system (for the general theory see [6]). There is a canonical length function $l : W \to \mathbf{N}$. Let q be a formal parameter, and let $\mathcal{H}_q(W, S)$ be the associative $\mathbf{C}$-algebra with basis $\{e_w\}_{w\in W}$ and relations

$$(e_s+1)(e_s-q) = 0 \qquad \text{if } s \in S$$
$$e_w e_{w'} = e_{ww'} \quad \text{if } l(ww') = l(w) + l(w')$$

See [6, Exercises 22-25, Ch. IV §2]. This is the Iwahori-Hecke algebra. If

$$\rho : \mathcal{H}_q(W, S) \to \mathrm{End}(V)$$

is a representation on a finite-dimensional vector space V over $\mathbf{C}$, we can define a matrix-valued formal series:

$$L(W, S, \rho, u) = \sum_{w\in W} \rho(e_w)u^{l(w)}.$$

When $q = 1$, $\mathcal{H}_q(W, S)$ is the ordinary group-ring of W over $\mathbf{C}$. If in addition, $\rho = 1$, $L(W, S, \rho, u)$ is the generating function that counts the elements in W of a given length relative to S. Special cases of these series were studied by a number of people, starting with Bott, who came upon them in the study of the loop spaces of Lie groups (see [22]), but it was Gyoja ([11], [12]) who saw the importance of including a nontrivial representation in the definition.

More generally, for any subset $I \subset S$, we have a subgroup $W_I \subset W$ generated by I, and we can define $L(W_I, S, \rho, u)$ as before, but with summation only over $w \in W_I$. Also, one can take for q a collection of numbers q_s, $s \in S$, with $q_s = q_{s'}$ if s and s' are W-conjugate. A set u of corresponding variables are introduced with the same property and $u^{l(w)}$ is defined as $u_{s_1} u_{s_2} ... u_{s_n}$ for a reduced decomposition of $w = s_1...s_n$. Also, the Iwahori-Hecke algebra can be defined with respect to the system $q = \{q_s\}$.

Gyoja proved:

Theorem 3.1. *$L(W, S, \rho, u)$ is a rational matrix-valued function of u.*

In fact there are simple recursion formulas for computing these, and explicit evaluations for $\rho = 1$. That these functions are rational is also a consequence of the fact that a Coxeter group has an automatic structure. Gyoja has also investigated the zeros and poles of $\det L(W, S, \rho, u)$ as well as functional equations satisfied by these in some cases.

Coxeter systems arise from Tits systems (G, B, N, S). The Weyl group of the system, defined as $W = N/B \cap N$ is a Coxeter group, with S identified with a subset of W. The key fact is the Bruhat decomposition:

$$G = \coprod_{w \in W} C(w), \quad C(w) = BwB.$$

There are two cases of interest: W is finite, and W is affine. The first case arises from taking G the rational points of a semisimple algebraic group over a field K; B is a Borel subgroup, N the normalizer of a maximal split torus. The second case is similar, but now K is a nonarchimedean local field, and B is an Iwahori subgroup. In the second case, B is a compact subgroup, and taking a Haar measure μ on G that assigns a mass of 1 to B, define $\mathcal{H}(G, B)$ to be the convolution algebra of locally constant, compactly-supported B-biinvariant complex-valued functions on G. Iwahori and Matsumoto proved, [20]:

Theorem 3.2. *Let G be the group of K-rational points of a semisimple and simply connected algebraic group over a nonarchimedean local field K. There is an isomorphism $C(w) \to e_w$ from $\mathcal{H}(G, B)$ to $\mathcal{H}_q(W, S)$, where W is the Weyl group of the Tits system, and $q_s = \mu(BsB)$ = number of right B-cosets in BsB.*

Remark. The result is true with appropriate modifications of the statement, for the K-points of a reductive group.

Example: $G = \mathrm{SL}_2(K)$ where K is a nonarchimedean local field. $O_K = O$ the ring of integers and π a generator of its maximal ideal $\mathfrak{p}$. q the cardinality of $O/\mathfrak{p}$.

$$B = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,\ d \in\ O^\times,\ b \in O,\ c \in \mathfrak{p},\ ad - bc = 1 \right\}$$

$N = N(K)$ is the group of monomial matrices. $T = T(K)$ is the group of diagonal matrices. $S = \{s_1,\ s_2\}$, where

$$s_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad s_2 = \begin{pmatrix} 0 & -1/\pi \\ \pi & 0 \end{pmatrix}$$

$q_{s_1} = q_{s_2} = q$, and the building X is a $q+1$-regular tree: the vertices are in 1-1 correspondence with the lattices (rank 2 O-submodules L of $V = K^2$) modulo homothety $L \sim xL$, all $x \in K^\times$. These fall into two orbits under G, the stabilizers corresponding to the two conjugacy classes of maximal compact subgroups $U = U_1 = B \cup Bs_1B = \mathrm{SL}_2(O)$ and $U_2 = B \cup Bs_2B$. The affine Weyl group is generated by s_1 and s_2, and is the infinite dihedral group. We have ($T_i = Bs_iB$)

$$\mathcal{H}(G,B) = \mathbf{C}[T_1,T_2]_{nc}, \quad T_i^2 - (q-1)T_i - q = 0$$

The convolution algebra of complex-valued U-biinvariant functions of G is the (usual) Hecke algebra. Letting $T = Us_1s_2U$, we have

$$\mathcal{H}(G,U) = \mathbf{C}[T]$$

and there is an injection $\mathcal{H}(G,U) \to \mathcal{H}(G,B)$.

Let $\Gamma \subset G = \mathrm{SL}_2(K)$ be a discrete cocompact subgroup. If $M \subset G$ is a compact subgroup, we denote by $C(\Gamma\backslash G/M)$ the space of functions $f : G \to \mathbf{C}$ such that

$$f(\gamma g m) = f(g) \quad \text{for all } \gamma \in \Gamma,\ m \in M.$$

Now $C(\Gamma\backslash G/M)$ is a right module for the convolution algebra $\mathcal{H}(G,M)$.

Recall that $Y = \Gamma\backslash X$ is a finite $(q+1)$-regular graph. It is bipartite: there are two equivalence classes of vertices whose stabilizer subgroups are U_1 and U_2 respectively. Therefore, we have an isomorphism:

$$C^0(X) = C(\Gamma\backslash G/U_1) \oplus C(\Gamma\backslash G/U_2).$$

where $C^0(X) = \mathrm{Hom}(C_0(X), \mathbf{C})$ is the group of 0-cochains of the graph. The group G acts transitively on the set of geometric edges of the tree X, thus the group of geometric 1-cochains of the graph is

$$C^1(X) = C(\Gamma\backslash G/B).$$

Note that $\dim C^1(X) = r_1$, whereas in Bass's proof we had $\dim C_1(X) = 2r_1$, because the edges there came with two possible orientations. $C^1(X)$ is a right module for the Iwahori-Hecke algebra $\mathcal{H}(G,B)$; let us denote the corresponding representation by ρ_Γ.

Proposition 3.3. *Let $\Gamma \subset \mathrm{SL}(2,K)$ be a discrete, cocompact and torsion-free subgroup, where K is a nonarchimedean local field. Let X be the Bruhat-Tits tree for $\mathrm{SL}(2,K)$, and $X_\Gamma = \Gamma\backslash X$ the corresponding graph. In the notation introduced above,*

$$\frac{\det L(W,S,\rho_\Gamma,u)}{\det L(W_1,S,\rho_\Gamma,u)\det L(W_2,S,\rho_\Gamma,u)} = Z(X_\Gamma,u)$$

where W_i is the subgroup generated by s_i.

Proof. Hashimoto [14, Main Theorem(I), pp. 224-225] showed that

$$Z(X_\Gamma,u) = \det(1 - u^2\rho^*(T_2T_1))^{-1}$$

in the notations of *loc. cit.* See also the discussion in [14, pp. 248-253] and [16, pp. 176-178] linking this to the Tits system. Note that later on that page Hashimoto redefines the zeta function so that the u^2 term becomes u (compare with formula 0.15 in his paper). In our notation

$$\rho^*(T_2T_1) = \rho_\Gamma(e_{s_2}e_{s_1}) = \rho_\Gamma(e_{s_2})\rho_\Gamma(e_{s_1})$$

the last equation holding because $e_{s_2s_1} = e_{s_2}e_{s_1}$, since $2 = l(s_2s_1) = l(s_2) + l(s_1)$. On the other hand, Gyoja evaluated his L-function for the infinite dihedral group, [11, Example 2.6, p. 97]. The result is, for any representation ρ:

$$L(W,S,\rho,u) = (1 + u\rho(e_{s_1}))(1 - u^2\rho(e_{s_2})\rho(e_{s_1}))^{-1}(1 + u\rho(e_{s_2}))$$

It is easy to see that

$$L(W_i, S, \rho, u) = 1 + u\rho(e_{s_i})$$

for $i = 1, 2$, so the proposition follows by taking determinants. □

Remarks. (1.) Note that $s_2 s_1$ is a generator of a canonical subgroup of W, namely the subgroup of translations.
(2.) In fact, Hashimoto's results apply to discrete cocompact and torsion-free subgroups of the K-rational points G of any semisimple simply-connected group over K of rational rank 1. In this case the building is also a tree, but it is biregular in general: in the above notation we have $q_{s_1} = q^{d_1}$ and $q_{s_2} = q^{d_2}$ are powers of the residual characteristic, possibly different, so we have a (q^{d_1}, q^{d_2})-regular tree. Hence, proposition 3.3 holds in this case as well.

4. Higher-dimensional buildings

Let G be the group of K-rational points of a semi-simple, simply connected algebraic group $\mathbf{G}$ over a nonarchimedean local field K. There is a contractible cell complex X of dimension l (= split rank of G) on which G acts, called the Bruhat -Tits building. General properties of these are summarized in [4] and [10]. There is an associated Tits system such that the cardinality of S is $l+1$. If $\Gamma \subset G$ is a discrete, cocompact subgroup, then $X_\Gamma = \Gamma \backslash X$ is a finite polyhedron, with a polysimplicial structure, which is a simplicial structure if $\mathbf{G}$ is simple. By a theorem of Borel and Harder [3], such Γ's exist, at least if $\mathrm{char}(K) = 0$. Assume that $\mathbf{G}$ is simple. The cells X of various dimensions are in bijective correspondence with the cosets G/B_I, where

$$B_I = BW_IB, \quad I \subset S$$

are the standard parahoric subgroups. Thus the cells of maximal dimension (chambers) are in one orbit and in a bijection with G/B. The vertices are in general in several orbits, corresponding to the maximal compact subgroups

$$U_s = B_{S-s}, \quad s \in S.$$

If Γ acts without fixed points on X, then X_Γ is a $K(\Gamma, 1)$-space. If that is so then

$$H^i(\Gamma, V) = H^i(X_\Gamma, \tilde{V}).$$

Here, $\varphi : \Gamma \to \mathrm{Aut}(V)$ is a representation, and $\tilde{V}$ is the corresponding sheaf on the space X_Γ. Assume V is a finite-dimensional $\mathbf{C}$-vector space. If in addition, $\mathbf{G}$ is simple over K and φ is a unitary representation, Garland showed that these cohomology groups are 0 for all $0 < i < l$ at least if the cardinality of the residue field of K is large enough, [5], [10].

For each $I \subset S$, let $C^I(X_\Gamma, V)$ be the space of all functions

$$f : G \longrightarrow V, \text{ such that } f(\gamma g b) = \rho(\gamma) f(g),\ \gamma \in \Gamma,\ b \in B_I.$$

These are finite-dimensional $\mathbf{C}$-vector spaces. In the special case where $\rho = 1$, this is just the vector space spanned by the cells of type I in X_Γ. The space of cochains of dimension r is

$$C^r(X_\Gamma, V) = \bigoplus_{\#I = l-r} C^I(X_\Gamma, V)$$

and these form a cochain complex. Assume again that φ is a unitary representation, then we get metrics on these cochain groups in a canonical way, and therefore Laplacians Δ_i (see Garland's paper). The spectra of these Laplacians are fundamental invariants of the group Γ and its representation φ.

Each $C^I(X_\Gamma, V)$ is a right $\mathcal{H}(G, B_I)$-module under convolution. In particular, taking $I = \emptyset$, we have a finite-dimensional representation ρ_Γ of the Iwahori-Hecke algebra $\mathcal{H}(G, B)$ on the space of cells of maximal dimension $C^\emptyset(X_\Gamma, V) = C^l(X_\Gamma, V)$ and thus a matrix $L(W', S, \rho_\Gamma, u)$ associated to it, for any subset $W' \subset W$ (for instance the subgroups W_I). The coefficients of the powers of t in the characteristic polynomial

$$\det(t - L(W', S, \rho_\Gamma, u))$$

and in particular, $\det L(W', S, \rho_\Gamma, u)$ are also fundamental invariants of the discrete subgroup Γ and its module V. Very little in known about these rational functions of u, but in the case of graphs, the relation given by proposition 3.3 suggests that their study may prove rewarding. For instance

$$\prod_{I \subset S} \det L(W_I, S, \rho_\Gamma, u)^{(-1)^{\# I}}$$

generalizes the expression in that proposition.

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