PLANAR ROBOTS AND THE INVERSE KINEMATIC PROBLEM: AN APPLICATION OF GROEBNER BASES

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ABSTRACT. We describe an idealized class of robots whose movements are fixed in a single plane. We describe and discuss the Forward Kinematic Problem and Inverse Kinematic Problem as it relates to such robots. We show that these problems can essentially be reduced to solving systems of polynomial equations. We describe and develop the relevant components of Groebner Bases and Elimination Theory. Using these tools, we analyze the geometry of two specific planar robots. We also define kinematic singularities in terms of the Jacobian matrix and investigate possible singularities for our example robots.

1. Introduction

The development of Groebner bases by Buchberger in the 1960s, and in particular the Buchberger algorithm, gives a systematic way for solving systems of polynomial equations. Consequently, many applications for Groebner bases have been discovered. One such application occurs in robotic kinematics, the study of robot motion.

In this paper, we will consider kinematics problems for a specific class of robots whose motions are restricted to a single plane. In the next section, we will show how such problems can be modeled in algebraic geometry as systems of polynomial equations. Afterwards, we will summarize the relevant concepts and results from the theory of Groebner bases. Finally, we will look at two specific robots and try to understand their geometry using the setup and methods from the rest of the paper.

2. Planar Robots

In this section we will give a geometric description of a robotic "arm". We will consider an idealized type of robot embedded in the standard Cartesian plane, called a planar robot. We will restrict our attention to arms composed of rigid segments and joints connected in series, like those in a human arm. One end of the arm will be fixed in position, while the other will be allowed to move through the plane by changing the settings of the various joints. We will call this end of a robotic arm the "hand". Our main goals are to consider various positions and orientations of the hand and to determine how we can manipulate the robot to reach a desired configuration. This is called the Inverse Kinematic Problem.

Actual robots are constructed using a variety of joint types. In our planar robots we will consider two types of joints. A planar revolute joint allows a segment to rotate through the plane relative to the previous segment. We describe the setting of such a joint as the counterclockwise angle θ between the first segment and next one. A prismatic

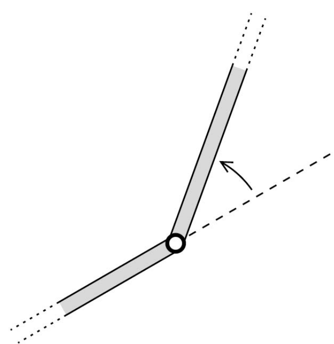

Figure 1. a revolute joint

joint permits the translation of one segment of the arm along an axis via retraction and extension of the joint. We describe the setting of a prismatic joint as the length by which it is extended. Thus, a given prismatic joint will have setting l such that $0 \leq l \leq m$, where m is the maxiumum length of the joint.

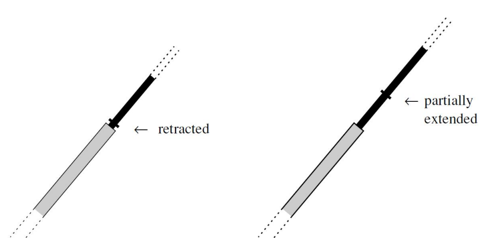

Figure 2. a prismatic joint

In a given robot, we may choose different rotation angles for the revolute joints and different lengths for the prismatic joints to place the hand at some position (a, b) in the plane. In the same way, we may manipulate the hand's orientation to be parallel to some unit vector $\mathbf{u}$ in $\mathbb{R}^2$. This observation that different joint settings cause different positions and orientations for the hand prompts the following definitions.

Definition 2.1. The *joint space* $\mathcal{J}$ for a planar robot cosisting of a series composed of segments joined by r revolute joints and p prismatic joints is defined as

$$\mathcal{J} = \overbrace{S^1 \times S^1 \times \ldots \times S^1}^{r \text{ times}} \times I_1 \times I_2 \times \ldots \times I_p, \tag{1}$$

where there is one S^1 factor for each revolute joint and the I_p represent the possible settings for each prismatic joint.

Definition 2.2. The *configuration space* $\mathcal{C}$ for a planar robot is defined as

$$\mathcal{C} = \mathcal{U} \times \mathcal{V}, \tag{2}$$

where $\mathcal{U} \subset \mathbb{R}^2$ consists of the subset of the plane whose points (a, b) represent possible hand positions, and where $\mathcal{V} = S^1$ represent possible hand orientations.

Each collection of joint settings will place the hand in some uniquely determined position, with some uniquely determined orientation. Therefore, we may define a function f from $\mathcal{J}$ to $\mathcal{C}$ which describes the hand configuration obtained by a specific collection of joint settings.

We can now formally describe the Inverse Kinematic Problem as follows: Given $c \in \mathcal{C}$, how can we determine one or all the $j \in \mathcal{J}$ such that $f(j) = c$?

To solve the Inverse Kinematic Problem, we must precisely describe the function f which maps joint settings to configurations. In other words, we need to describe the hand's position and orientation in terms of the rotation settings of the revolute joints and the lengths of the prismatic joints.

For both of the robots we will consider, we fix the first revolute joint at the center of the standard Cartesian coordinate system, which we designate (x_1, y_1). We then introduce a local coordinate system at each revolute joint to describe the relative positions of the segments which meet at that joint. At each revolute joint i, we introduce an (x_{i+1}, y_{i+1}) coordinate system with origin at joint i. The positive x_{i+1} axis lies along the direction of segment $i+1$, and the positive y_{i+1} axis is determined such that it forms a standard right-handed coordinate system.

We may transform coordinates in the (x_{i+1}, y_{i+1}) coordinate system to the (x_i, y_i) coordinate system. To do this, we rotate a coordinate by the angle θ_i and then translate by the vector $(l_i, 0)$. The rotation alligns the x_{i+1} axis with the x_i axis, and the translation moves the origin of the (x_{i+1}, y_{i+1}) coordinate system to coincide with the origin of the (x_i, y_i) coordinate system.

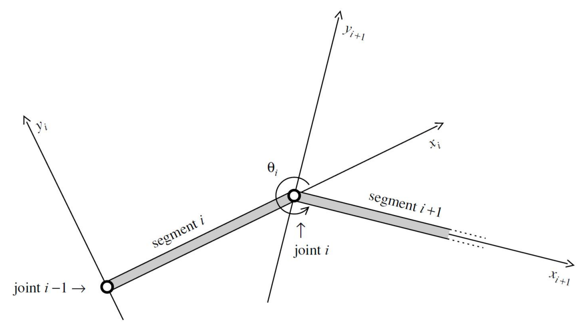

FIGURE 3. Local coordinate systems

Therefore, for a given point p in the (x_1, y_1) coordinate system, we may transform its (x_{i+1}, y_{i+1}) coordinates to its (x_i, y_i) coordinates as follows:

$$\begin{pmatrix} a_i \\ b_i \end{pmatrix} = \begin{pmatrix} \cos\theta_i & -\sin\theta_i \\ \sin\theta_i & \cos\theta_i \end{pmatrix} \cdot \begin{pmatrix} a_{i+1} \\ b_{i+1} \end{pmatrix} + \begin{pmatrix} l_i \\ 0 \end{pmatrix} \tag{3}$$

We will generally use a common shorthand for this coordinate transformation which combines the rotation and translation into single matrix:

$$\begin{pmatrix} a_i \\ b_i \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\theta_i & -\sin\theta_i & l_1 \\ \sin\theta_i & \cos\theta_i & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a_{i+1} \\ b_{i+1} \\ 1 \end{pmatrix} = A_i \cdot \begin{pmatrix} a_{i+1} \\ b_{i+1} \\ 1 \end{pmatrix} \tag{4}$$

One final tool we will use to analyze our robots is the Jacobian matrix.

Definition 2.3. The Jacobian matrix for a function $f : \mathcal{J} \to \mathcal{C}$ is

$$J_f\left(j_1, j_2, \ldots, j_n\right) = \begin{pmatrix} \frac{\delta f_1}{\delta j_1} & \cdots & \frac{\delta f_n}{\delta j_n} \\ \frac{\delta f_2}{\delta j_1} & \cdots & \frac{\delta f_n}{\delta j_n} \\ \frac{\delta f_3}{\delta j_1} & \cdots & \frac{\delta f_n}{\delta j_n} \end{pmatrix}, \tag{5}$$

where f_i is the i-th component function of f.

The Jacobian matrix defines the best linear approximation of f. Thus, f and J_f should behave similarly for a particular joint configuration in $\mathcal{J}$ We will say that the rank of the Jacobian matrix is the maximal number of linear independent columns (or rows) in the matrix.

We will assign dimensions to both the joint space $\mathcal{J}$ and the configuration space $\mathcal{C}$. Intuitively, we can understand the dimension of each space to represent the number of degrees of freedom in that space space. For example, the configuration space has dimension 3.

If the rank of the Jacobian matrix is smaller than the dimensions of $\mathcal{J}$ and $\mathcal{C}$, then the robot's behavior at that point differs from what we would expect.

Definition 2.4. A kinematic singularity for a robot is a point $j \in \mathcal{J}$ such that $J_f(j)$ has rank strictly less than $\min(\dim(\mathcal{J}),\ \dim(\mathcal{C}))$.

3. Groebner Bases and Polynomial Equations

In this section we will discuss some necessary terms and concepts from the theory of Groebner bases. We will begin by defining monomials, polynomials, and some relevant terminology.

Definition 3.1. A monomial in $x_1, \ldots, x_n$ is a product of the form

$$x_1^{\alpha_1} \cdot x_2^{\alpha_2} \cdots x_n^{\alpha_n}, \tag{6}$$

where all of the exponents $\alpha_1, \ldots, \alpha_n$ are nonnegative integers. The total degree of this monomial is the sum $\alpha_1 + \cdots + \alpha_n$.

Definition 3.2. A polynomial f in $x_1, \ldots, x_n$ with coefficients in k is a finite linear combination (with coefficients in k) of monomials. We will write a polynomial f in the form

$$f = \sum_{\alpha} a_\alpha x^\alpha, a_\alpha \in k, \tag{7}$$

where the sum is over a finite number of n-tuples $\alpha = (\alpha_1, \ldots, \alpha_n)$. The set of all polynomials in $x_1, \ldots, x_n$ with coefficients in k is denoted $k[x_1, \ldots, x_n]$.

Definition 3.3. Let $f = \sum_\alpha a_\alpha x^\alpha$ be a polynomial in $k[x_1, \ldots, x_n]$.

i. We call a_α the coefficient of the monomial x^α.
ii. If $a_\alpha \neq 0$, then we call $a_\alpha x^\alpha$ a term of f.
iii. The total degree of f, denoted *deg* (f), is the maximum $|\alpha|$ such that the coefficient a_α is nonzero.

In polynomials of one variable, we generally order the monomials by degree. However, the order in which we list the monomials for a polynomial in general is not as straightforward.

Definition 3.4. A **monomial ordering** on $k > [x_1, \dots, x_n]$ is any relation $>$ on $\mathbb{Z}^n_{\geq 0}$ satisfying:

i. $>$ is a total ordering on $\mathbb{Z}^n_{\geq 0}$.
ii. If $\alpha > \beta$ and $\gamma \in \mathbb{Z}^n_{\geq 0}$, then $\alpha + \gamma > \beta + \gamma$.
iii. $>$ is a well-ordering on $\mathbb{Z}^n_{\geq 0}$.

The monomial ordering we are interested in for solving systems of polynomial equations is called lexicographic order. It is similar to the alphabetic ordering of words in a dictionary.

Definition 3.5 (Lexicographic Order)**.** Let $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{Z}^n_{\geq 0}$. We say $\alpha >_{lex} \beta$ if, in the vector difference $\alpha - \beta \in \mathbb{Z}^n$, the leftmost nonzero entry is positive.

One can easily show that lexicographic order is a monomial ordering on $k > [x_1, \dots, x_n]$.

We will now introduce some important ideas from the theory of polynomial rings which we will use to develop a definition of a Groebner basis.

Definition 3.6. Let $f_1, \dots, f_s$ be polynomials in $k\,[x_1, \dots, x_n]$. Then we set

$$\langle f_1, \dots, f_s \rangle = \left\{ \sum_{i=1}^{s} h_i f_i, \text{ where } h_1, \dots, h_s \in k\,[x_1, \dots, x_n] \right\}. \quad (8)$$

Definition 3.7. A subset $I \subset k\,[x_1, \dots, x_n]$ is an ideal if it satisfies:

i. $0 \in I$.
ii. If $f, g \in I$, then $f + g \in I$.
iii. If $f \in I$ and $h \in k\,[x_1, \dots, x_n]$, then $hf \in I$.

Proposition 3.8. *If $f_1, \dots, f_s \in k\,[x_1, \dots, x_n]$, then $\langle f_1, \dots, f_s \rangle$ is an ideal of $k\,[x_1, \dots, x_n]$. We will call $\langle f_1, \dots, f_s \rangle$ the ideal generated by $f_1, \dots, f_s$.*

Theorem 3.9 (Hilbert Basis Theorem)**.** *Every ideal $I \subset k\,[x_1, \dots, x_n]$ has a finite generating set. That is, $I = \langle g_1, \dots, g_t \rangle$ for some $g_1, \dots, g_t \in I$.*

If f is an element of $k[x_1, \dots, x_n]$, we denote $\textsc{lt}(f)$ as the leading term of this polynomial under a given monomial ordering.

Definition 3.10. Fix a monomial order. A finite subset $G = \{g_1, \dots, g_n\}$ of an nonzero ideal I is said to be a **Groebner basis** if

$$\langle \textsc{lt}(g_1), \dots, \textsc{lt}(g_n) \rangle = \langle \textsc{lt}(I) \rangle,$$

where the right-hand side is the ideal generated by the leading terms of every element of I.

Proposition 3.11. *Fix a monomial order. Then every nonzero ideal $I \subset k[x_1, \ldots, x_n]$ has a Groebner basis. Furthermore, any Groebner basis for an ideal I is a basis of I.*

4. Applications

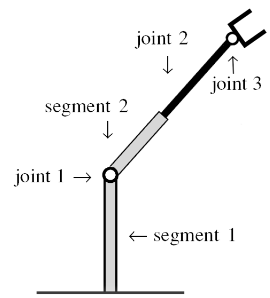

Figure 4. 2 revolute joints and 1 prismatic joint

Robot 1. The first robot we will consider is composed of two revolute joints and one prismatic joint. The first prismatic joint is fixed at the origin, and connects the fixed segment of the arm with the second segment. A prismatic joint with variable length is attached to the end of the second segment. Finally, the second revolute joint joins the prismatic joint on the second segment with the hand, which we will consider to be the third segment of the arm.

Both revolute joints are allowed to rotate freely in the plane, and the prismatic joint may be set at any length between a minimum length, m_1, and a maximum length, m_2. Therefore, the joint space is given by $\mathcal{J} = S_1 \times S_1 \times [m_1, m_2]$.

We would like to find a function $f : \mathcal{J} \to \mathcal{C}$ describes a rule for getting from a particular set of joint settings to the uniquely determined configuration for those settings. Let us suppose the length of the fixed segment to be 1. Given values for θ_1, θ_2, and l_2, basic trigonometry shows that the position of the hand is the point in $\mathbb{R}^2$ defined by $((1 + l_2)\cos(\theta_1), (1 + l_2)\sin(\theta_1))$. The orientation of the hand is given simply by the sum of the angles traversed by each individual revolute joint.

Thus, an explicit function $f : \mathcal{J} \to \mathcal{C}$ can be written in terms of trigonometric functions as

$$f(\theta_1, \theta_2, l_2) = \begin{pmatrix} (1 + l_2)\cos(\theta_1) \\ (1 + l_2)\sin(\theta_1) \\ \theta_1 + \theta_2 \end{pmatrix}. \tag{9}$$

With this function in hand, we can now solve the Inverse Kinematic Problem. In order to apply the techniques of Groebner Bases to the problem, we wish to replace these trigonometric functions with polynomial functions via the substitutions $c_i = \cos(\theta_i)$ and $s_i = \sin(\theta_i)$, where $c_i^2 + s_i^2 = 1$ for $i = 1, 2$.

For this particular robot, the orientation of the hand is trivial since the final revolute joint does not affect the position of the hand. Given any value of θ_1, a desired orientation $\alpha = \theta_1 + \theta_2$ is given by setting $\theta_2 = \alpha - \theta_1$. Therefore, we will need only consider the hand's position. Based on (9), one can see that the possible ways to reach a fixed position $(a, b) \in \mathbb{R}^2$ are represented by solving the system of equations

$$a = (1 + l_2)c_1 \tag{10}$$
$$b = (1 + l_2)s_1 \tag{11}$$
$$0 = s_1^2 + c_1^2 - 1 \tag{12}$$

for $s1$, $s2$, $c1$, and $c2$.

Fix a lexicographic monomial ordering with $c_1 > s_1 > l_2$. Using Mathematica, we find that

$$\{-a^2 - b^2 + (1 + l_2)^2, b(1 + l_2) - s_1(a^2 + b^2), c_1(a^2 + b^2) - a(1 + l_2)\}$$

forms a Groebner basis for the ideal generated by equations (10)-(12). Setting each polynomial in this Groebner basis equal to 0 and solving yields two sets of solutions:

$$c_1 = \frac{a}{\sqrt{a^2 + b^2}},\ s_1 = \frac{b}{\sqrt{a^2 + b^2}},\ l_2 = \sqrt{a^2 + b^2} - 1$$

$$c_1 = -\frac{a}{\sqrt{a^2 + b^2}},\ s_1 = -\frac{b}{\sqrt{a^2 + b^2}},\ l_2 = -\sqrt{a^2 + b^2} - 1.$$

The second solution may be discarded because it gives a negative value for l_2.

Given that l_2 can take on any value in $[0, m_3]$ for some fixed positive value of m_3, a and b can take on any values that satisfy the inequality $1 \leq \sqrt{a^2 + b^2} \leq m_3 + 1$, or equivalently $1 \leq a^2 + b^2 \leq (m_3 + 1)^2$. Thus, the possible placements in the configuration space trace out the annulus between the circles centered at the origin (that is, joint 1) with radii 1 and $m_3 + 1$.

Because $\min\{dim(J_f), dim(C)\} = 3$, a kinematic singularity occurs when the dimension of the Jacobian matrix evaluated at a point is less than or equal to 2. As the mapping is defined by (9), the Jacobian matrix is represented by

$$J_f(\theta_1, \theta_2, l_2) = \begin{pmatrix} -(1+l_2)\sin(\theta_1) & 0 & \cos(\theta_1) \\ (1+l_2)\cos(\theta_1) & 0 & \sin(\theta_1) \\ 1 & 1 & 0 \end{pmatrix}.$$

The dimension of J_f is less than 3 precisely when $\det(J_f) = 0$. We have

$$0 = \det(J_f) = (1+l_2)(\cos^2(\theta_1) + \sin^2(\theta_1)) = 1 + l_2$$

if and only if $l_2 = -1$, which is not possible because $l_2 \in [0, m_3]$. Thus, there are no points of singularity in the problem.

Robot 2. The second planar robot of interest is composed of two revolute joints, each attached to segments of length 1, followed by a third revolute joint that leads into a prismatic joint of length l_4, where $m_1 \leq l_4 \leq m_2$.

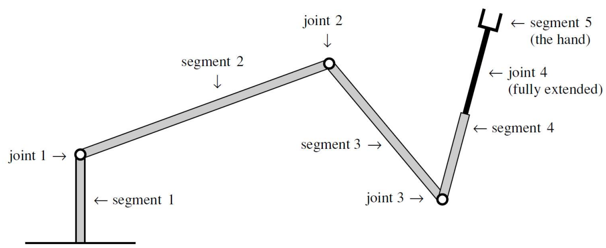

FIGURE 5. 3 revolute joints and 1 prismatic joint

As discussed earlier, the transformation from the $i+1^{\text{th}}$ coordinate system to the i^{th} coordinate system can be represented by the equation

$$\begin{pmatrix} a_i \\ b_i \\ 1 \end{pmatrix} = \begin{pmatrix} \cos(\theta_i) & -\sin(\theta_i) & l_i \\ \sin(\theta_i) & \cos(\theta_i) & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a_{i+1} \\ b_{i+1} \\ 1 \end{pmatrix} = A_i \cdot \begin{pmatrix} a_{i+1} \\ b_{i+1} \\ 1 \end{pmatrix}. \quad (13)$$

Because the origins of the (x_1, y_1) and (x_2, y_2) coordinate systems are both at joint 1, we say that $l_1 = 0$ so that

$$A_1 = \begin{pmatrix} \cos(\theta_1) & -\sin(\theta_1) & 0 \\ \sin(\theta_1) & \cos(\theta_1) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Bearing in mind that $l_2 = l_3 = 1$ and that the coordinates of the hand in the (x_4, y_4) coordinate system are $(l_4, 0)$, we can find the hand's location in terms of the global coordinate system by working back one joint at a time through matrix multiplication:

$$\begin{pmatrix} x_1 \\ y_1 \\ 1 \end{pmatrix} = A_1 A_2 A_3 \begin{pmatrix} l_4 \\ 0 \\ 1 \end{pmatrix}. \tag{14}$$

Multiplying and simplifying with sum and difference trigonometric identities yields

$$\begin{pmatrix} x_1 \\ y_1 \\ 1 \end{pmatrix} = \begin{pmatrix} l_4 \cos(\theta_1 + \theta_2 + \theta_3) + \cos(\theta_1 + \theta_2) + \cos(\theta_1) \\ l_4 \sin(\theta_1 + \theta_2 + \theta_3) + \sin(\theta_1 + \theta_2) + \sin(\theta_1) \\ 1 \end{pmatrix}. \tag{15}$$

This gives us the unique hand placement for any particular values of θ_1, θ_2, θ_3, and l_4. Furthermore, the orientation is once again given by the sum of the angles of the individual revolute joints. Hence, we can define a mapping from the joint space of this robot to its configuration space by

$$g(\theta_1, \theta_2, \theta_3, l_4) = \begin{pmatrix} l_4 \cos(\theta_1 + \theta_2 + \theta_3) + \cos(\theta_1 + \theta_2) + \cos(\theta_1) \\ l_4 \sin(\theta_1 + \theta_2 + \theta_3) + \sin(\theta_1 + \theta_2) + \sin(\theta_1) \\ \theta_1 + \theta_2 + \theta_3 \end{pmatrix}.$$

As before, we wish to convert these transcendental functions into polynomials in order to apply the techniques of Groebner bases to find solutions by setting $c_i = \cos(\theta_i)$ and $s_i = \sin(\theta_i)$, where $c_i^2 + s_i^2 = 1$ for $i = 1, 2, 3$.

The orientation of the hand cannot be expressed as a polynomial in c_i and s_i. We will show, however, that $c = \cos(\theta_1 + \theta_2 + \theta_3)$ and $s = \sin(\theta_1 + \theta_2 + \theta_3)$ can be written in this way. At the cost of introducing an additional component to the configuration space, we may use the values of c and s to find the uniquely determined angle. Using basic trigonometric identities, it can be seen that

$$c = c_3(c_1 c_2 - s_1 s_2) - s_3(c_1 s_2 + s_1 c_2)$$

and similarly

$$s = c_3(s_1 c_2 + c_1 s_2) + s_3(c_1 c_2 - s_1 s_2).$$

Given a fixed hand placement $(a, b) \in \mathbb{R}^2$ and orientation in terms of $c = \cos(\theta_1 + \theta_2 + \theta_3)$ and $s = \sin(\theta_1 + \theta_2 + \theta_3)$, we now wish to find the possible joint settings that yield this configuration.

We need to find solutions to the following system of polynomial equations:

$$-a + c_1 + c_1 c_2 - s_1 s_2 + l_4(-s_1(c_3 s_2 + c_2 s_3) + c_1(c_2 c_3 - s_2 s_3)) = 0$$
$$-b + s_1 + c_2 s_1 + c_1 s_2 + l_4(c_1(c_3 s_2 + c_2 s_3) + s_1(c_2 c_3 - s_2 s_3)) = 0$$

$-c + c_3(c_1c_2 - s_1s_2) - (c_2s_1 + c_1s_2)s_3 = 0$
$-d + c_3(c_2s_1 + c_1s_2) + (c_1c_2 - s_1s_2)s_3 = 0$
$-1 + c_1^2 + s_1^2 = 0$
$-1 + c_2^2 + s_2^2 = 0$
$-1 + c_3^2 + s_3^2 = 0$
$-1 + c^2 + d^2 = 0$

Using Mathematica, we obtain the following Groebner basis:
$-1 + c^2 + d^2,$
$-1 + c_3^2 + s_3^2,$
$b^2 - 2abcd + a^2d^2 - b^2d^2 + 2bcc_3s_2 - 2ac_3ds_2 + s_2^2 + 2bcs_3 - 2ads_3 + 2c_3s_2s_3,$
$bc - ad + c_3s_2 + s_3 + c_2s_3,$
$-bcc_3 + bcc_2c_3 + ac_3d - ac_2c_3d - s_2 + c_2s_2 + bcs_2s_3 - ads_2s_3 + 2s_2s_3^2,$
$-1-b^2+c_2^2+2abcd-a^2d^2+b^2d^2-2bcc_3s_2+2ac_3ds_2-2bcs_3+2ads_3-2c_3s_2s_3,$
$-b + acd - c_2c_3d + bd^2 + s_1 - cs_3 + ds_2s_3,$
$c_1 - cc_2c_3 + bcd - ad^2 + ds_3 + cs_2s_3,$
$-ac + c_3 + c_2c_3 - bd + l_4 - s_2s_3$

It is difficult to gain new information from this Groebner basis alone. Instead, we consider various specializations of the system. That is, we choose desired configurations and recompute the Groebner basis. We find that the edge of the disk of radius 4 has only one orientation possible, and that there is only one joint setting which gives that configuration. Inside the disk, more joint settings become possible at increasingly less restrictive orientations. In general, with a fixed prismatic setting, at least two joint settings will reach a valid configuration. On the disk of radius 1, all orientations become possible under some joint setting.

The Jacobian matrix for this robot, in reduced row-echelon form, is given by:

$$\begin{pmatrix} 1 & 0 & 0 & \cos\theta_3 \csc\theta_2 \\ 0 & 1 & 0 & -\cos\theta_3 \cot\frac{\theta_2}{2} + \sin\theta_3 \\ 0 & 0 & 1 & \cos(\theta_2+\theta_3)\csc\theta_2 \end{pmatrix}$$

Thus, there are no kinematic singularities for this robot because the rank of J_g is always 3.

Acknowledgements

Work on this project was made possible by the Louisiana State University Mathematics Department through the SMILE program, funded by a VIGRE grant from the National Science Foundation. The authors would like to thank Matthew Dawson and Dr. Robert Lax for their help and guidance on this project.

References

[1] Cox, D., Little, J., and O'Shea, D., *Ideals, Varieties, and Algorithms: An Introduction to Computational Algebraic Geometry and Commutative Algebra*, Spinger Undergraduate Texts in Mathematics, Springer, New York, 2008.

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