PHASE PLANE DIAGRAMS OF DIFFERENCE EQUATIONS

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Abstract. We will be determining qualitative features of a discrete dynamical system of homogeneous difference equations with constant coefficients. By creating phase plane diagrams of our system we can visualize these features, such as convergence, equilibrium points, and stability.

1. Introduction

Continuous systems are often approximated as discrete processes, meaning that we look only at the solutions for positive integer inputs. Difference equations are recurrence relations, and first order difference equations only depend on the previous value. Using difference equations, we can model discrete dynamical systems. The observations we can determine, by analyzing phase plane diagrams of difference equations, are if we are modeling decay or growth, convergence, stability, and equilibrium points.

For this paper, we will only consider two dimensional systems.

Suppose $x(k)$ and $y(k)$ are two functions of the positive integers that satisfy the following system of difference equations:

$$\begin{aligned} x(k+1) &= ax(k) + by(k) \\ y(k+1) &= cx(k) + dy(k). \end{aligned}$$

This system is more simply written in matrix from

$$\mathbf{z}(k+1) = A\mathbf{z}(k),$$

where $\mathbf{z}(k) = \begin{pmatrix} x(k) \\ y(k) \end{pmatrix}$ is a column vector and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. If $\mathbf{z}(0) = \mathbf{z}_0$ is the initial condition then the solution to the initial value problem

$$\mathbf{z}(k+1) = A\mathbf{z}(k), \ \mathbf{z}(0) = \mathbf{z}_0,$$

is

$$\mathbf{z}(k) = A^k \mathbf{z}_0.$$

Each $\mathbf{z}(k)$ is represented as a point $(x(k), y(k))$ in the Euclidean plane. The set of points $\{(x(k), y(k))\, t \in \mathbb{N}\}$ traces out a path. The direction of this path is determined by k increasing. The directed path is called a trajectory. By changing initial conditions, we can create different trajectories for the given system. The phase portrait is a diagram consisting of these multiple trajectories. The shape of paths, the direction of trajectories, and equilibrium solutions are some of the qualitative features we will explore.

The general solution may seem easy to solve, however, A^k will be difficult to compute for large values of k. We can use a tool called the affine transformation to change our variables and create a matrix J which will hopefully be diagonal or close to diagonal. Using the properties of affine transformations we will show any given matrix A is related to one of only three possible Jordan Canonical forms.

2. Affine Transformation

Affine transformations are important because certain shapes in the (u, v) phase plane are preserved in the (x, y) phase plane. While the solutions are not continuous, the trajectory can be resembled by a continuous function. More notably, if $\mathbf{z} = P\mathbf{z}$ is an affine transformation then

(1) a linear trajectory in the (u, v) phase plane is transformed to a linear trajectory in the (x, y) phase plane. If the liner solutions in the (u, v) phase plane goes through the origin, so does the transformed linear solutions.

(2) an elliptical trajectory in the (u, v) phase plane is transformed to an elliptical trajectory in the (x, y) phase plane.

(3) a spiral shaped trajectory in the (u, v) phase plane is transformed to a spiral shaped trajectory in the (x, y) phase plane.

(4) a power curve shaped trajectory in the (u, v) phase plane is transformed to a power curve shaped trajectory in the (x, y) phase plane, for example, parabolas and hyperbolas.

(5) a tangent line L to a curve C in the (u, v) phase plane is transformed to the tangent like $P(L)$ to the curve $P(C)$ in the (x, y) phase plane.

(6) a curved trajectory C that lies in a region R in the (u, v) phase plane is transformed to a curved trajectory $P(C)$ that lies in the region $P(R)$ in the (x, y) phase plane.

The goal is to find an affine transformation P such that $J = P^{-1}AP$ is particularly simple; specifically, one of the three forms described in Section 3.

Two matrices A and J are similar if there exists an invertible matrix P such that $J = P^{-1}AP$. We say the phase portraits of $\mathbf{z}(k+1) = A\mathbf{z}(k)$ and $\mathbf{z}(k+1) = J\mathbf{z}(k)$ are affine equivalent if A and J are similar.

To study the phase portrait of $\mathbf{z}(k+1) = A\mathbf{z}$ we can consider the affine transformation $\mathbf{w}(k+1) = J\mathbf{w}$, where J is a 2×2 matrix that has a particularly simple form. Let P be a 2×2 invertible matrix. The change in variables

$$\mathbf{z} = P\mathbf{w} \tag{2.1}$$

is called the affine transformation. Specifically, if

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}, \ \mathbf{z} = \begin{bmatrix} x \\ y \end{bmatrix}, \ \textit{and } \mathbf{w} = \begin{bmatrix} u \\ v \end{bmatrix}$$

then the equation $\mathbf{z} = P\mathbf{w}$ becomes

$$\begin{aligned} x &= p_{11}u + p_{12}v \\ y &= p_{21}u + p_{22}v. \end{aligned}$$

Since P is invertible we also have $\mathbf{w} = P^{-1}\mathbf{z}$. We are now able to go from one set of variables to the other. Since P is a constant matrix, we have

$$\mathbf{w}(k+1) = P^{-1}\mathbf{z}(k+1) = P^{-1}A\mathbf{z} = P^{-1}AP\mathbf{w}. \tag{2.2}$$

Setting $J = P^{-1}AP$, Equation 2.2 becomes

$$\mathbf{w}(k+1) = J\mathbf{w} \tag{2.3}$$

and the associated initial condition becomes $\mathbf{w}_0 = \mathbf{w}(0) = P^{-1}\mathbf{z}_0$. Once $\mathbf{w}$ is determined we can find $\mathbf{z}$ using the equation $\mathbf{z} = P\mathbf{w}$.

3. Jordan Canonical Form

For every matrix A we find the similar matrix J through the affine transformation. Using the following theorem we know that J will be diagonal or close to diagonal, and therefore much easier to compute J^k for any k.

Theorem 3.1. *Let A be a two by two real matrix. Then there is a nonsingular real matrix P so that*

$$A = PJP^{-1},$$

where:

(1) *If A has real eigenvalues λ_1, λ_2, not necessarily distinct, with linearly independent eigenvectors, then*

$$J_1 = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.$$

(2) *If A has a single eigenvalue λ with a single independent eigenvector, then*

$$J_2 = \begin{bmatrix} \lambda & 0 \\ 1 & \lambda \end{bmatrix}.$$

(3) *If A has complex eigenvalues $\alpha \pm i\beta$, then*

$$J_3 = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}.$$

These are the only three simple forms of J. Now that we have a simple matrix J, we can begin creating our phase plane diagrams for analysis.

4. Analysis

The simplest solution, or trivial solution is when $x = 0$ and $y = 0$ simultaneously. This trivial solution is represented by at point at the origin in the phase plane. The trivial solution is called a critical, or equilibrium, point. A non trivial solution to the system traces a smooth curve which we called a trajectory. We take a sampling of trajectories to create our phase plane diagram. The geometrical properties of the phase plane diagram are related to the algebraic characteristics of the matrix A, which are preserved through the affine transformation. We can see that the eigenvalues of A play a decisive role in determining many of important characteristics of the phase portrait. We will now analyze the possible combinations of eigenvalues with initial values.

Case 1:

$$J_1 = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.$$

General solution: $\mathbf{w}(k) = \begin{pmatrix} c_1\lambda_1^k \\ c_2\lambda_2^k \end{pmatrix}$

When $0 < \lambda_1 < \lambda_2 < 1$, If $c_1 = 0$, then as k approaches infinity, $\mathbf{w}(k)$ approaches zero, along the v-axis. If $c_2 = 0$, then as k approaches infinity, $\mathbf{w}(k)$ approaches zero, along the u-axis. When c_1 is non-zero and c_2 is non-zero, the points lie along the line

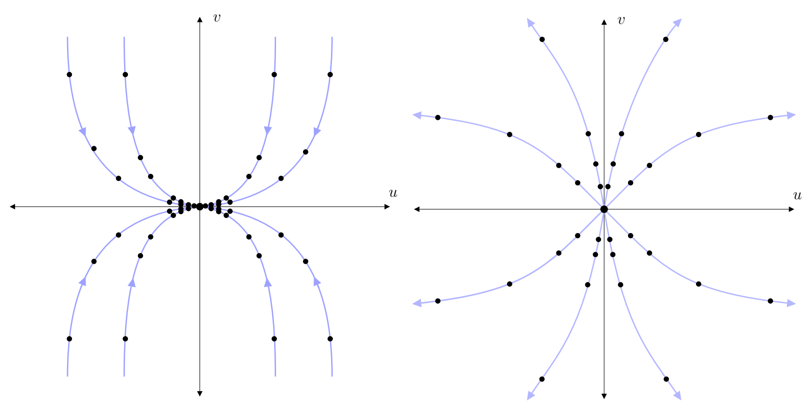
$$v(k) = c_2\lambda_2^{\frac{ln(\frac{u}{c_1})}{ln\lambda_1}}.$$

In this instance, all trajectories approach zero. The diagram will resemble 1a in Figure 1. This is called a sink, or stable node.

When $\lambda_1 > \lambda_2 > 1$, If $c_1 = 0$, then as k approaches infinity, $\mathbf{w}(k)$ diverges to infinity, along the v-axis. If $c_2 = 0$, then as k approaches infinity, $\mathbf{w}(k)$ diverges to infinity, along the u-axis. When c_1 is non-zero and c_2 is non-zero, the points lie along the line

$$v(k) = c_2\lambda_2^{\frac{ln(\frac{u}{c_1})}{ln\lambda_1}}.$$

In this instance, all trajectories diverge to infinity. The diagram looks like 1b in Figure 1. This is called a source, or unstable node.



(A) Sink (B) Source

FIGURE 1. J_1 Diagrams

When $\lambda_1 = 1$ and $\lambda_2 < \lambda_1$, then $\mathbf{w}(k) = c_1u(k) + c_2\lambda_2v(k)$. The points will trace a vertical line at $u = c_1$, tending towards $v = 0$. The equilibrium points will fill out the whole u-axis.

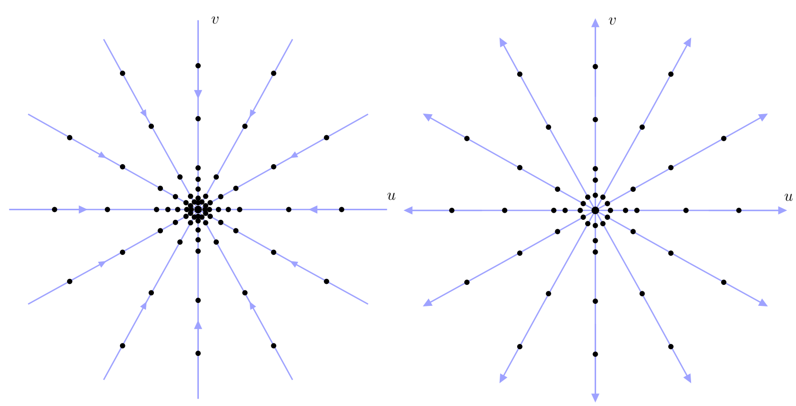
This is called a degenerate node.

When $\lambda_1 < -1$ and $\lambda_2 > 1$, then if $c_1 = 0$, as k approaches infinity, $\mathbf{w}(k)$ diverges to infinity, along the v-axis. If $c_2 = 0$, then as k approaches infinity, $\mathbf{w}(k)$ diverges to infinity, along the u-axis. For $c_1 \neq 0$ and $c_2 \neq 0$ when k is even, the points lie along the curve in quadrant I and IV. When k is odd, the points lie along the curve in quadrant II and III. For all curves, the points diverge to infinity.

This is called a source with reflection.

When $0 < \lambda_1 = \lambda_2 < 1$, if $c_1 = 0$, then as k approaches infinity, $\mathbf{w}(k)$ approaches zero, along the v-axis. If $c_2 = 0$, then as k approaches infinity, $\mathbf{w}(k)$ approaches zero, along the u-axis. When $c_1 \neq 0$ and $c_2 \neq 0$ the points lie alone the line $v(k) = \frac{c_2}{c_1}u(k)$, and the slope depends on the initial conditions c_1 and c_2. The diagram looks like Figure 2a. This is called a stable star node.

When $\lambda_1 = \lambda_2 > 1$, if $c_1 = 0$, then as k approaches infinity, $\mathbf{w}(k)$ diverges to infinity, along the v-axis. If $c_2 = 0$, then as k approaches infinity, $\mathbf{w}(k)$ diverges to infinity, along the u-axis. When $c_1 \neq 0$ and $c_2 \neq 0$ the points lie alone the line $v(k) = \frac{c_2}{c_1}u(k)$, and the slope depends on the initial conditions c_1 and c_2. The diagram looks like Figure 2b. This is called an unstable star node.

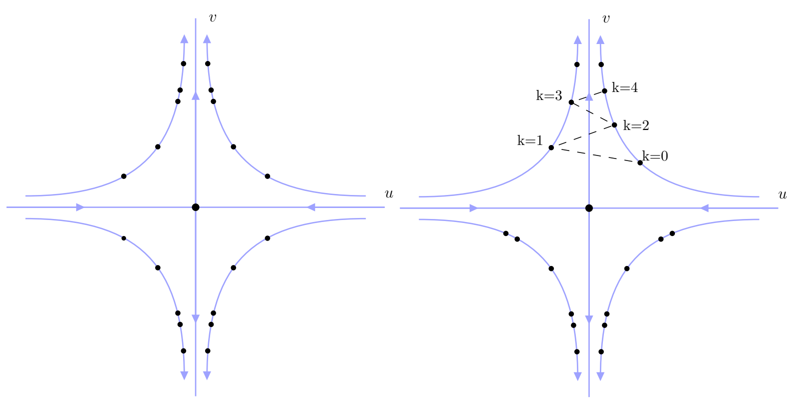


(A) Stable Star (B) Unstable Star

FIGURE 2. J_1 Diagrams

When $0 < \lambda_1 < 1$ and $\lambda_2 > 1$, if $c_1 = 0$ and $c_2 = 0$ we get the point $(0, 0)$. If $c_1 = 0$, then as k approaches infinity, $\mathbf{w}(k)$ diverges to infinity, along the v-axis. If $c_2 = 0$, then as k approaches infinity, $\mathbf{w}(k)$ tends towards zero, along the u-axis. When $c_1 \neq 0$ and $c_2 \neq 0$ the points lie alone the line $v(k) = c_2\lambda_2^{ln(\frac{u(k)}{c_1})ln\lambda_1}$. The diagram looks like Figure 3a. This is called a saddle.

When $-1 < \lambda_1 < 0 < 1 < \lambda_2$, if $c_1 = 0$ and $c_2 = 0$ we get the point $(0, 0)$. If $c_1 = 0$, then as k approaches infinity, $\mathbf{w}(k)$ diverges to infinity, along the v-axis. If $c_2 = 0$, then as k approaches infinity, $\mathbf{w}(k)$ tends towards zero, along the u-axis. For $c_1 \neq 0$ and $c_2 \neq 0$ when k is even, the points lie along the curve in quadrant I and IV, which resemble a reflection over the u-axis. When k is odd, the points lie along the curve in quadrant II and III, which resemble a reflection over the u-axis. For all non-zero initial conditions, the points diverge to infinity. In general, the points lie alone the line $v(k) = c_2\lambda_2^{ln(\frac{u(k)}{c_1})ln\lambda_1}$. The diagram looks like Figure 3b This is called a saddle with reflection.



(A) Saddle

(B) Saddle with Reflection

FIGURE 3. J_1 Diagrams

Case 2:

$$J_2 = \begin{bmatrix} \lambda & 0 \\ 1 & \lambda \end{bmatrix}.$$

General solution: $\mathbf{w}(k) = \lambda^{k-1} \begin{pmatrix} c_1\lambda \\ c_1k + c_2\lambda \end{pmatrix}$

For $0 < |\lambda| < 1$ we can observe that:

When $c_1 = 0$, then $\mathbf{w}(k)$ tends towards zero along the v-axis as k approaches infinity. Specifically, when $c_1 = 0, c_2 < 0$, then $\mathbf{w}(k)$ moves along the negative v-axis, and when $c_1 = 0, c_2 > 0$, then $\mathbf{w}(k)$ moves along the positive v-axis.

If $c_1 \neq 0$, then as k approaches infinity, $\mathbf{w}(k)$ tends towards zero. The shape of trajectories can be represented by the function

$$v = \frac{u}{|\lambda|} \log_{|\lambda|} \frac{u}{c_1} + u\frac{c_2}{c_1}.$$

It is important to note that when $-1 < \lambda < 0$, the points oscillate across the v-axis. Specifically, the even k values trace the trajectories on the positive half of the u-axis, and the odd k values trace the trajectories on the negative half of the u-axis. The diagram will look like Figure 4a.

For $|\lambda| = 1$ we can observe that:

If $c_1 = 0$, $\mathbf{w}(k)$ diverges to infinity along the v-axis as k approaches infinity. Specifically, if $c_1 = 0, c_2 < 0$, $\mathbf{w}(k)$ moves along the negative v-axis, and when $c_1 = 0, c_2 > 0$, $\mathbf{w}(k)$ moves along the positive v-axis.

If $c_1 \neq 0$, $\mathbf{w}(k)$ diverges to infinity as k approaches infinity. The trajectories can be represented by the function

$$u = c_1.$$

It is important to note that when $\lambda = -1$ the points oscillate across the u-axis. Specifically, the even k values trace the trajectories on the positive half of the v-axis, and the odd k values trace the trajectories on the negative half of the v-axis. The diagram will look like Figure 4b.

Finally, when $|\lambda| > 1$ we can observe that:

If $c_1 = 0$, $\mathbf{w}(k)$ diverges to infinity along the v-axis as k approaches infinity. If $c_1 \neq 0$, then $\mathbf{w}(k)$ diverges to infinity as k approaches infinity. The trajectories can be represented by the function

$$v = \frac{u}{|\lambda|} \log_{|\lambda|} \frac{u}{c_1} + u\frac{c_2}{c_1}.$$

It is important to note that when $\lambda < -1$, the points oscillate across the v-axis. Specifically, the even k values trace the trajectories on the positive half of the u-axis, and the odd k values trace the trajectories on the negative half of the u-axis. The diagram will look like Figure 4c.

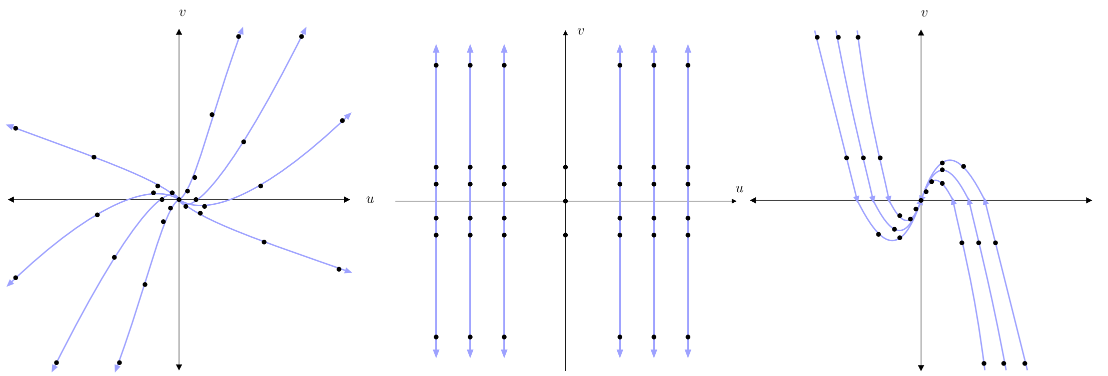


FIGURE 4. J_2 Diagrams

Case 3:

$$J_3 = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}.$$

General solution: $\mathbf{w}(k) = |\lambda|^k \begin{pmatrix} \cos k\theta & \sin k\theta \\ -\sin k\theta & \cos k\theta \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

where $|\lambda| = \sqrt{\alpha^2 + \beta^2}$ and $\theta = \tan^{-1}(\frac{\beta}{\alpha})$

If $\alpha^2 + \beta^2 = 1$ the points lie on a circle, rotating counterclockwise. Under this condition, J is called the rotation matrix. The diagram looks like Figure 5a.

When $\alpha^2 + \beta^2 > 1$ the phase plane diagram will be an unstable spiral, meaning it will diverge from the origin towards infinity. The diagram looks like Figure 5b.

When $\alpha^2 + \beta^2 < 1$ the phase plane diagram will be a stable spiral, that is, tending towards the origin. The diagram looks like Figure 5c.

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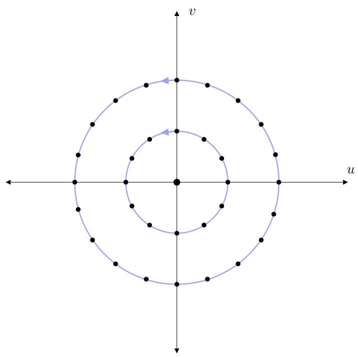
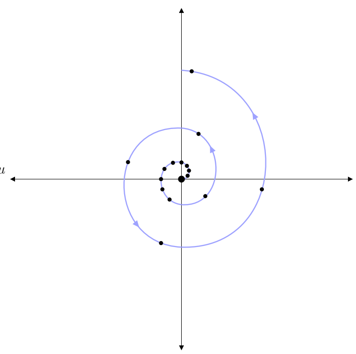
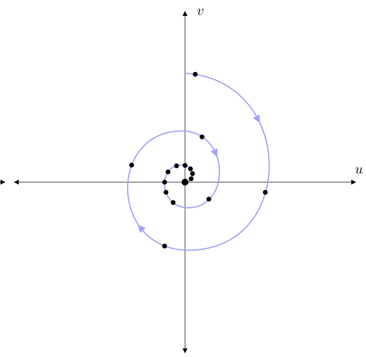


(A) Circle



(B) Unstable Spiral



(C) Stable Spiral

FIGURE 5. J_3 Diagrams

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