Phase Plane Diagrams of Difference Equations

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Phase Plane Diagrams of Difference Equations

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## Outline

- **Introduction**
  - Terminology
  - Affine Transformation

- **Phase Plane Diagrams**
  - Jordan Canonical Forms

- **Example**

- **Conclusion**
Goals

- Model discrete dynamical systems to determine outcome.
- Determine qualitative features of a system of homogeneous difference equations with constant coefficients.
System of Difference Equations

\[ x(k + 1) = ax(k) + by(k) \]
\[ y(k + 1) = cx(k) + dy(k) \]

General solution:
\[ z(k) = A^k z(0) \]

Where \( z = \begin{pmatrix} x(k) \\ y(k) \end{pmatrix} \) and \( A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \)
• Each solution is in the set
  \[ \{(x(k), y(k)) : k \in \mathbb{N}\} \]
• Trajectory
• Phase Plane Diagram
Affine Transformation

- A tool for changing variables
- Preserves collinearity
Change in Variables

\[ z(k) = Pw(k) \]
\[ w = \begin{bmatrix} u(k) \\ v(k) \end{bmatrix}, \quad P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \]

Create \( J = P^{-1}AP \)
gives \( w(k+1) = Jw(k) \)
General solution: \( w(k) = J^k w(0) \)
\[ x = p_{11} u + p_{12} v \]
\[ y = p_{21} u + p_{22} v. \]
Let $A$ be a two by two real matrix. Then there is a nonsingular real matrix $P$ so that

$$A = PJP^{-1},$$

where:

If $A$ has real eigenvalues $\lambda_1, \lambda_2$, not necessarily distinct, with linearly independent eigenvectors, then

$$J_1 = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.$$
Jordan Canonical Form Theorem cont.

If $A$ has a single eigenvalue $\lambda$ with a single independent eigenvector, then

$$J_2 = \begin{bmatrix} \lambda & 0 \\ 1 & \lambda \end{bmatrix}.$$ 

If $A$ has complex eigenvalues $\alpha \pm i\beta$, then

$$J_3 = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}.$$
Spectral Radius Theorem

\[ r(A) = \max \{|\lambda| : \lambda \text{ is an eigenvalue}\} \]

If \( r(A) < 1 \), then any solution to \( z(k) = A^k z(0) \) has the property
\[ \lim_{k \to \infty} A^k z(0) = 0. \]

If \( r(A) \geq 1 \), some solutions \( z(k) \) does not tend toward the origin as \( k \to \infty \).
Case 1

\[ J_1 = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \]

Where \( \lambda_1, \lambda_2 \in \mathbb{R} \)

General solution: \( \mathbf{w}(k) = \begin{pmatrix} c_1 \lambda_1^k \\ c_2 \lambda_2^k \end{pmatrix} \)
Source and Sink

Figure: Trajectories

(c) $\lambda_1 > \lambda_2 > 1$  
(d) $0 < \lambda_1 < \lambda_2 < 1$
Source and Sink

(a) $\lambda_1 > \lambda_2 > 1$

(b) $0 < \lambda_1 < \lambda_2 < 1$

Figure: $v = cu^p$
Unstable and Stable Star

(a) $\lambda_1 = \lambda_2 > 1$

(b) $0 < \lambda_1 = \lambda_2 < 1$

Figure: $v = \frac{c_2}{c_1} u$
Saddle and Saddle with Reflection

(a) $0 < \lambda_1 < 1, \lambda_2 > 1$

(b) $-1 < \lambda_1 < 0 < 1 < \lambda_2$

Figure: $v = cu^p$
Degenerate Node

\[ w(k) = \begin{pmatrix} c_1 \\ \lambda^k c_2 \end{pmatrix} \]

\[ 0 < \lambda_2 < \lambda_1 = 1 \]

Figure: $0 < \lambda_2 < \lambda_1 = 1$
Case 2

\[ J_2 = \begin{bmatrix} \lambda & 0 \\ 1 & \lambda \end{bmatrix} \]

Where \( \lambda \in \mathbb{R} \)

General solution: \( w(k) = \lambda^{k-1} \begin{pmatrix} c_1 \lambda \\ c_1 k + c_2 \lambda \end{pmatrix} \)
**Introduction**

**Phase Plane Diagrams**

**Example**

**Conclusion**

**Jordan Canonical Forms**

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**Figure**:

\[ v = \frac{u}{|\lambda|} \log |\lambda| \frac{u}{c_1} + u \frac{c_2}{c_1} \]

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Phase Plane Diagrams of Difference Equations
Case 3

\[ J_3 = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} \]

Where \( \alpha, \beta \in \mathbb{R} \)

General solution: \( w(k) = |\lambda|^k \begin{pmatrix} \cos k\theta & \sin k\theta \\ -\sin k\theta & \cos k\theta \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \) where

\[ |\lambda| = \sqrt{\alpha^2 + \beta^2} \text{ and } \theta = \tan^{-1}\left( \frac{\beta}{\alpha} \right) \]
(a) $\alpha^2 + \beta^2 = 1$

(b) $\alpha^2 + \beta^2 > 1$

(c) $\alpha^2 + \beta^2 < 1$

Figure: $J_3$ Diagrams
Example

\[ x(k + 1) = x(k) + y(k) \]
\[ y(k + 1) = 0.25x(k) + y(k) \]

Here \( A = \begin{bmatrix} 1 & 1 \\ 0.25 & 1 \end{bmatrix} \)

The eigenvalues of \( A \) are \( \lambda_1 = \frac{1}{2} \) and \( \lambda_2 = \frac{3}{2} \).

\[ P = \begin{bmatrix} 2 & -2 \\ 1 & 1 \end{bmatrix} \] and \( J = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{3}{2} \end{bmatrix} \)
(a) Phase Plane of $A$

(b) Phase Plane of $J$
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Thank You!