Phase Plane Diagrams of Difference Equations

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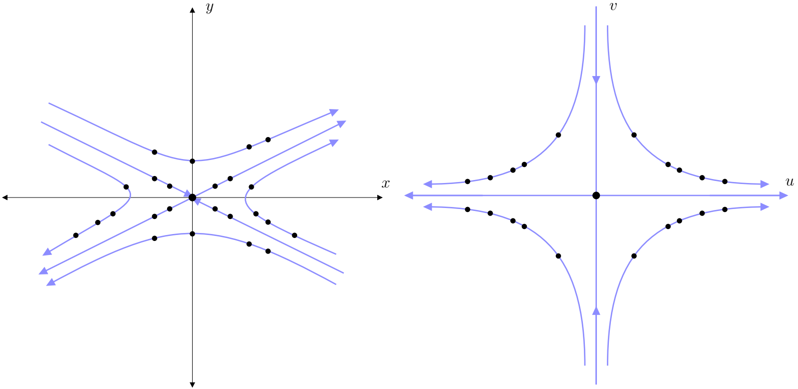

y
x
v
u

Outline

Goals

- Model discrete dynamical systems to determine outcome.
- Determine qualitative features of a system of homogeneous difference equations with constant coefficients.

System of Difference Equations

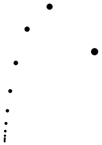
$$x(k+1) = ax(k) + by(k)$$
$$y(k+1) = cx(k) + dy(k)$$

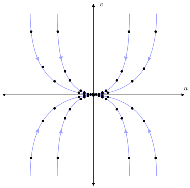
General solution:

$$\mathbf{z}(k) = A^k \mathbf{z}(0)$$

Where $z = \begin{pmatrix} x(k) \\ y(k) \end{pmatrix}$ and $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

- Each solution is in the set $\{(x(k), y(k)) : k \in \mathbb{N}\}$
- Trajectory
- Phase Plane Diagram





Affine Transformation

- A tool for changing variables
- Preserves collinearity

Change in Variables

$$\mathbf{z}(k) = P\mathbf{w}(k)$$

$$\mathbf{w} = \begin{bmatrix} u(k) \\ v(k) \end{bmatrix},\ P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$$

Create $J = P^{-1}AP$
gives $\mathbf{w}(k+1) = J\mathbf{w}(k)$
General solution: $\mathbf{w}(k) = J^k\mathbf{w}(0)$

$$x = p_{11}u + p_{12}v$$

$$y = p_{21}u + p_{22}v.$$

Jordan Canonical Form Theorem

Let A be a two by two real matrix. Then there is a nonsingular real matrix P so that

$$A = PJP^{-1},$$

where:

If A has real eigenvalues λ_1, λ_2, not necessarily distinct, with linearly independent eigenvectors, then

$$J_1 = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.$$

Jordan Canonical Form Theorem cont.

If A has a single eigenvalue λ with a single independent eigenvector, then

$$J_2 = \begin{bmatrix} \lambda & 0 \\ 1 & \lambda \end{bmatrix}.$$

If A has complex eigenvalues $\alpha \pm i\beta$, then

$$J_3 = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}.$$

Spectral Radius Theorem

$r(A) = \max\{|\lambda| : \lambda \text{ is an eigenvalue}\}$

If $r(A) < 1$, then any solution to $\mathbf{z}(k) = A^k\mathbf{z}(0)$ has the property

$$\lim_{k\to\infty} A^k z(0) = 0.$$

If $r(A) \geq 1$, some solutions $\mathbf{z}(k)$ does not tend toward the origin as $k \to \infty$.

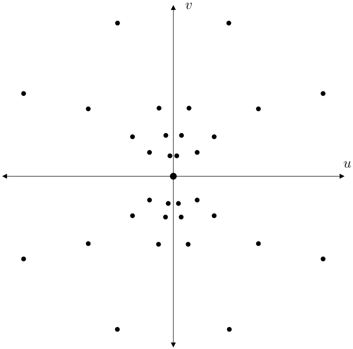
Case 1

$$J_1 = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

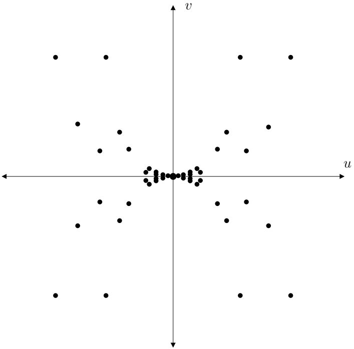
Where $\lambda_1, \lambda_2 \in \mathbb{R}$

General solution: $\mathbf{w}(k) = \begin{pmatrix} c_1 \lambda_1^k \\ c_2 \lambda_2^k \end{pmatrix}$

Source and Sink



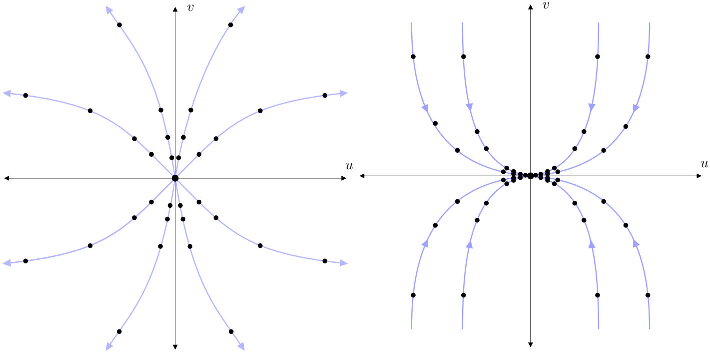
(c) $\lambda_1 > \lambda_2 > 1$



(d) $0 < \lambda_1 < \lambda_2 < 1$

Figure: Trajectories

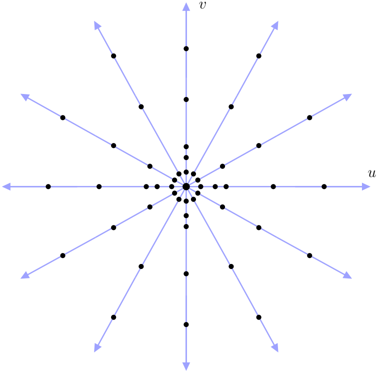
Source and Sink



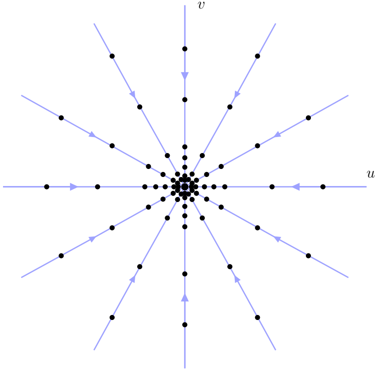
(a) $\lambda_1 > \lambda_2 > 1$ (b) $0 < \lambda_1 < \lambda_2 < 1$

Figure: $v = cu^p$

Unstable and Stable Star



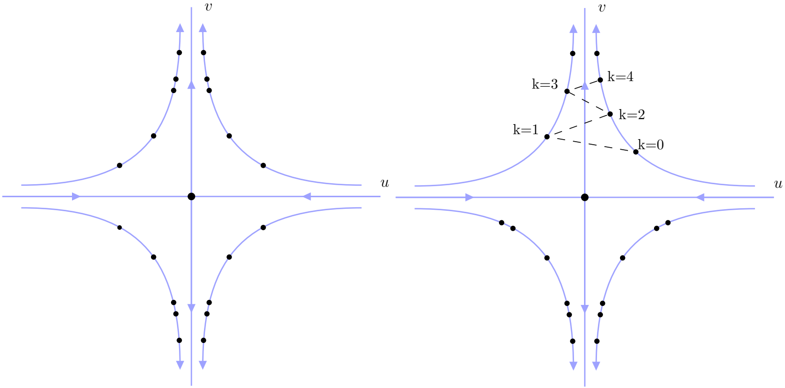
(a) $\lambda_1 = \lambda_2 > 1$



(b) $0 < \lambda_1 = \lambda_2 < 1$

Figure: $v = \frac{c_2}{c_1} u$

Saddle and Saddle with Reflection



(a) $0 < \lambda_1 < 1, \lambda_2 > 1$

(b) $-1 < \lambda_1 < 0 < 1 < \lambda_2$

Figure: $v = cu^p$

Degenerate Node

$$\mathbf{w}(k) = \begin{pmatrix} c_1 \\ \lambda_2^k c_2 \end{pmatrix}$$

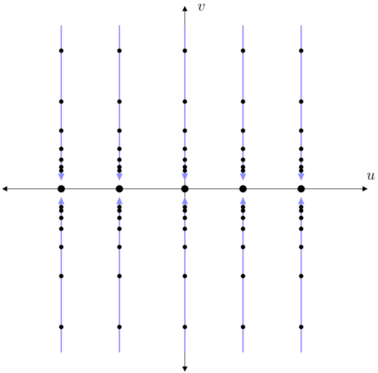


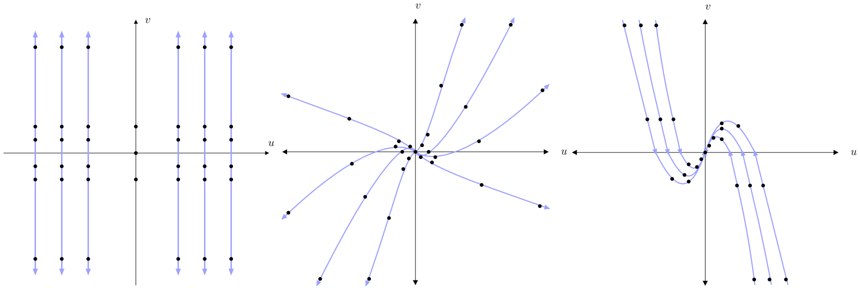
Figure: $0 < \lambda_2 < \lambda_1 = 1$

Case 2

$$J_2 = \begin{bmatrix} \lambda & 0 \\ 1 & \lambda \end{bmatrix}$$

Where $\lambda \in \mathbb{R}$

General solution: $\mathbf{w}(k) = \lambda^{k-1} \begin{pmatrix} c_1 \lambda \\ c_1 k + c_2 \lambda \end{pmatrix}$



(a) $|\lambda| = 1$ (b) $|\lambda| > 1$ (c) $|\lambda| < 1$

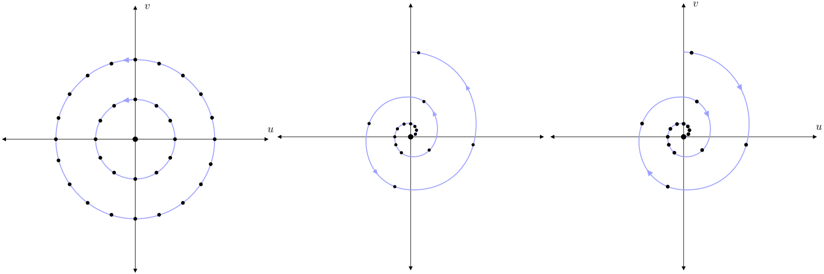
Figure: $v = \frac{u}{|\lambda|} \log_{|\lambda|} \frac{u}{c_1} + u\frac{c_2}{c_1}$

Case 3

$$J_3 = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$$

Where $\alpha, \beta \in \mathbb{R}$

General solution: $\mathbf{w}(k) = |\lambda|^k \begin{pmatrix} \cos k\theta & \sin k\theta \\ -\sin k\theta & \cos k\theta \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ where $|\lambda| = \sqrt{\alpha^2 + \beta^2}$ and $\theta = \tan^{-1}(\frac{\beta}{\alpha})$



(a) $\alpha^2 + \beta^2 = 1$ (b) $\alpha^2 + \beta^2 > 1$ (c) $\alpha^2 + \beta^2 < 1$

Figure: J_3 Diagrams

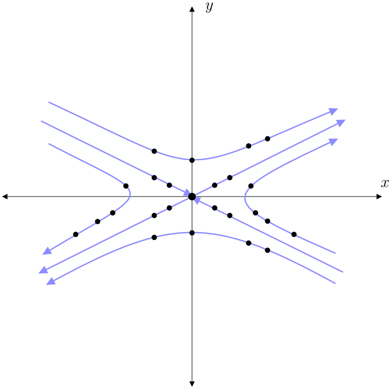
Example

$$x(k+1) = x(k) + y(k)$$
$$y(k+1) = 0.25x(k) + y(k)$$

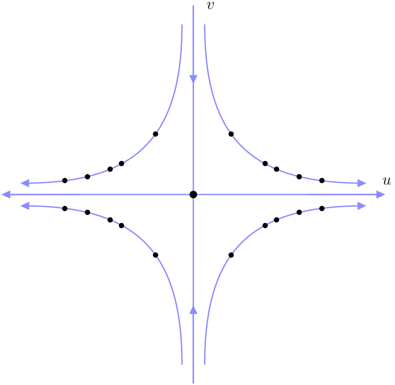
Here $A = \begin{bmatrix} 1 & 1 \\ 0.25 & 1 \end{bmatrix}$

The eigenvalues of A are $\lambda_1 = \frac{1}{2}$ and $\lambda_2 = \frac{3}{2}$.

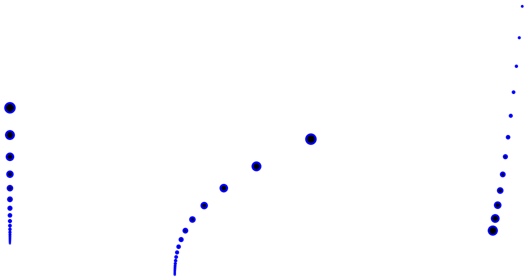
$P = \begin{bmatrix} 2 & -2 \\ 1 & 1 \end{bmatrix}$ and $J = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{3}{2} \end{bmatrix}$



(a) Phase Plane of A



(b) Phase Plane of J



Thank You!