Optimal Lower Bounds on the Stress and Strain Fields Inside Random Two-Phase Composites

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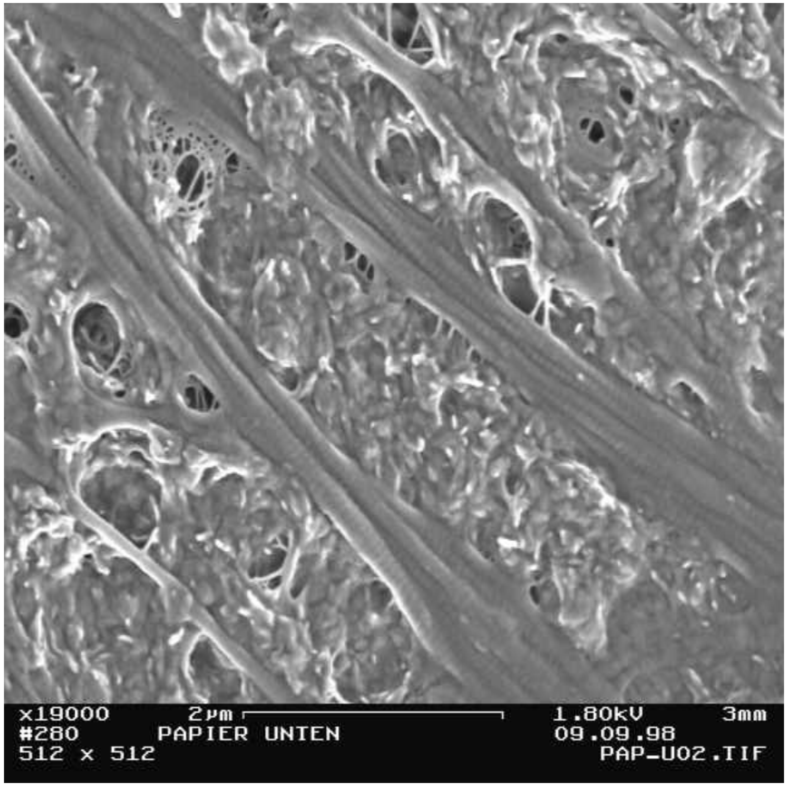
Outline

- Introduction
  - Load transfer between different length scales in composites and failure initiation
  - Higher order moments of the local stress and strain fields
- Elastic boundary value problem for composite materials
- Optimal lower bounds

Load transfer

- Many composite structures are hierarchical in nature and are made up of substructures distributed across several length scales. Examples include fiber reinforced laminates as well as naturally occurring structures like bone.
- From the perspective of failure initiation it is crucial to quantify the load transfer between length scales. It is common knowledge that the load transfer can result in local stresses that are significantly greater than the applied macroscopic stress.



Trabecular Bone

Moments of the local fields

- Quantities useful for the study of load transfer include the higher order moments of the local stress (strain). The higher moments are sensitive to local stress concentrations generated by the interaction between the microstructure and the macroscopic load.
- Optimal lower bounds on the maximum and the higher order moments of the local stress (strain) inside random composites made from two isotropic elastic materials in prescribed proportions.

$$< \chi_1 |\sigma(\mathbf{x})|^r >^{1/r} \quad \text{and} \quad < \chi_2 |\epsilon(\mathbf{x})|^r >^{1/r} \text{ for } 2 \leq r \leq \infty$$

$$\|\sigma\|_{L^\infty(Q)} = \lim_{r \to \infty} < |\sigma(\mathbf{x})|^r >^{1/r}$$

Elastic boundary value problem

The elastic stress and strain fields $\sigma(\mathbf{x})$ and $\epsilon(\mathbf{x})$ satisfy

$$\epsilon_{ij}(\mathbf{x}) = \frac{\partial_j u_i(\mathbf{x}) + \partial_i u_j(\mathbf{x})}{2} \qquad \text{and} \qquad \sigma(\mathbf{x}) = C(\mathbf{x})\epsilon(\mathbf{x}).$$

where $\mathbf{u}$ is the displacement field and C is the local elasticity tensor

$$C(\mathbf{x}) = \chi_1(\mathbf{x})C^1 + \chi_2(\mathbf{x})C^2$$

The two materials are isotropic

$$C^i = 2\mu_i \mathbf{\Lambda}^s + d\kappa_i \mathbf{\Lambda}^h, \text{ for } i = 1, 2,$$

where μ_i and κ_i are the shear and bulk moduli of the i-th material. The shear and hydrostatic projection tensors $\mathbf{\Lambda}^s$ and $\mathbf{\Lambda}^h$ are given by

$$\begin{aligned} \Lambda^h_{ijkl} &= \frac{1}{d}\delta_{ij}\,\delta_{kl}, \\ \Lambda^s_{ijkl} &= \frac{1}{2}\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) - \frac{1}{d}\delta_{ij}\,\delta_{kl}. \end{aligned}$$

Elastic boundary value problem

- The equation of elastic equilibrium inside each phase is given by

$$\operatorname{div} \sigma = 0$$

- Continuity of the displacement $\mathbf{u}$ and the traction $\sigma \mathbf{n}$, $\mathbf{n}$ being the unit normal, across the interface

$$\mathbf{u}_{|_1} = \mathbf{u}_{|_2} \quad \text{and} \quad \sigma \mathbf{n}_{|_1} = \sigma \mathbf{n}_{|_2}$$

- It is assumed that the composite is periodic with period cell Q (unit square or cube).
- The composite is subjected to an applied macroscopic stress $\overline{\sigma} = \langle \sigma \rangle$ or macroscopic strain $\overline{\epsilon} = \langle \epsilon \rangle$.
- The effective elastic tensor C^e relates the average stress to the average strain and is given by

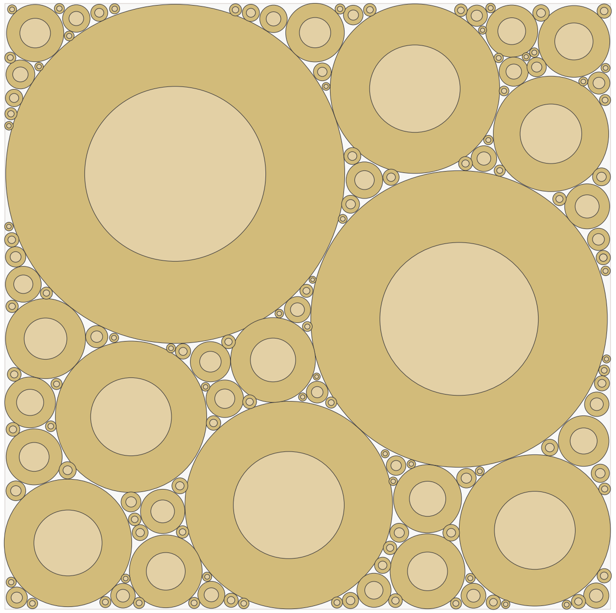
$$\overline{\sigma} = C^e \overline{\epsilon}.$$

Hydrostatic Applied Stress

Optimal lower bounds on the moments of the local stress:
The stress field inside material-1 satisfies

$$\langle \chi_1 |\sigma(\mathbf{x})|^r \rangle^{1/r} \;\geq\; \theta_1^{1/r} \frac{\kappa_1\kappa_2 + 2\frac{d-1}{d}\mu_2\kappa_1}{\kappa_1\kappa_2 + 2\frac{d-1}{d}\mu_2\left(\theta_1\kappa_1 + \theta_2\kappa_2\right)} \left|\overline{\sigma}\right|, \text{ for } 2 \leq r \leq \infty.$$

Optimality: For $d = 2(3)$, the local stress inside material-1 for the coated cylinder (sphere) assemblage with core of material-1 and coating of material-2 attains the lower bound for every r in $[2, \infty]$.



Hydrostatic Applied Stress

Optimal lower bounds on the moments of the local stress:
The stress field inside material-2 satisfies

$$\langle \chi_2 |\sigma(\mathbf{x})|^r \rangle^{1/r} \ \geq \ \theta_2^{1/r} \frac{\kappa_1\kappa_2 + 2\frac{d-1}{d}\mu_2\kappa_2}{\kappa_1\kappa_2 + 2\frac{d-1}{d}\mu_2\left(\theta_1\kappa_1 + \theta_2\kappa_2\right)} \, |\overline{\sigma}| \, , \ \text{for } 2 \leq r \leq \infty.$$

Moreover, for $d = 2(3)$, the local stress inside material-2 for the coated cylinder (sphere) assemblage with core of material-2 and coating of material-1 attains the lower bound for every r in $[2, \infty]$.

The stress field inside the composite satisfies (assuming $\kappa_1 > \kappa_2$)

$$\||\sigma(\mathbf{x})|\|_{L^\infty(Q)} \ \geq \ \frac{\kappa_1\kappa_2 + 2\frac{d-1}{d}\mu_2\kappa_1}{\kappa_1\kappa_2 + 2\frac{d-1}{d}\mu_2\left(\theta_1\kappa_1 + \theta_2\kappa_2\right)} \, |\overline{\sigma}| \, .$$

Moreover, for $d = 2(3)$, the stress field inside the coated cylinder (sphere) assemblage with core of material-1 and coating of material-2 attains the lower bound.

Equal bulk moduli composite

For a composite in which the bulk moduli are the same across the two phases ($\kappa_1 = \kappa_2 = \kappa$), the stress field inside the i-th material for which $\mu_i = \mu_+$ satisfies

$$\langle \chi_i |\mathbf{\Lambda}^s \sigma(\mathbf{x})|^r \rangle^{1/r} \;\geq\; \theta_i^{1/r} \left| \mathbf{\Lambda}^s \overline{\sigma} \right|, \text{ for } 2 \leq r \leq \infty$$

The stress field inside the composite satisfies

$$\| |\mathbf{\Lambda}^s \sigma(\mathbf{x})| \|_{L^\infty(Q)} \;\geq\; \left| \mathbf{\Lambda}^s \overline{\sigma} \right|,$$

and

$$\| |\sigma(\mathbf{x})| \|_{L^\infty(Q)} \;\geq\; \left| \overline{\sigma} \right|.$$

Moreover for $d = 2$ these lower bounds are optimal. These bounds hold for $d = 3$.

Derivation of the lower bound

- From Jensen's inequality

$$\left\langle \chi_1 |\mathbf{\Lambda}^s \sigma(\mathbf{x})|^2 \right\rangle \;\; \geq \frac{1}{\theta_1} \left|\langle \chi_1 \mathbf{\Lambda}^s \sigma(\mathbf{x})\rangle\right|^2 \tag{1}$$

- The shear component of the stress in material-1 satisfies

$$\begin{aligned} \langle \chi_1 \mathbf{\Lambda}^s \sigma(\mathbf{x})\rangle &= \langle \chi_1 \mathbf{\Lambda}^s C(\mathbf{x})\epsilon(\mathbf{x})\rangle \\ &= 2\mu_1 \langle \chi_1 \mathbf{\Lambda}^s \epsilon(\mathbf{x})\rangle \end{aligned} \tag{2}$$

- The shear component of the average stress can be written as

$$\begin{aligned} \mathbf{\Lambda}^s \overline{\sigma} = \langle \mathbf{\Lambda}^s \sigma(\mathbf{x})\rangle &= \langle \mathbf{\Lambda}^s C(\mathbf{x})\epsilon(\mathbf{x})\rangle \\ &= \left\langle \mathbf{\Lambda}^s (C^2 + \chi_1 (C^1 - C^2))\epsilon(\mathbf{x})\right\rangle \\ &= 2\mu_2 \mathbf{\Lambda}^s \overline{\epsilon} + 2(\mu_1 - \mu_2)\langle \chi_1 \mathbf{\Lambda}^s \epsilon(\mathbf{x})\rangle \end{aligned} \tag{3}$$

Derivation of the lower bound

- Thus from (2) and (3), we have

$$\langle \chi_1 \mathbf{\Lambda}^s \sigma(\mathbf{x}) \rangle = \frac{2\mu_1\mu_2}{\mu_1 - \mu_2} \left(\frac{1}{2\mu_2} \mathbf{\Lambda}^s \overline{\sigma} - \mathbf{\Lambda}^s \overline{\epsilon} \right) \tag{4}$$

- Since $\kappa_1 = \kappa_2$

$$\mathbf{\Lambda}^s \overline{\epsilon} = \mathbf{\Lambda}^s (C^e)^{-1} \overline{\sigma} = (C^e)^{-1} \mathbf{\Lambda}^s \overline{\sigma} \tag{5}$$

- Using Eqs. (4) and (5), inequality (1) becomes

$$\begin{aligned} \langle \chi_1 |\mathbf{\Lambda}^s \sigma(\mathbf{x})|^2 \rangle &\geq \frac{1}{\theta_1} \left| \frac{2\mu_1\mu_2}{\mu_1 - \mu_2} \left(\frac{1}{2\mu_2} \mathbf{\Lambda}^s \overline{\sigma} - (C^e)^{-1} \mathbf{\Lambda}^s \overline{\sigma} \right) \right|^2 \\ &\geq \frac{1}{\theta_1} \left(\frac{2\mu_1\mu_2}{\mu_1 - \mu_2}\right)^2 \frac{\left(\frac{1}{2\mu_2} \mathbf{\Lambda}^s \overline{\sigma} : \mathbf{\Lambda}^s \overline{\sigma} - ((C^e)^{-1} \mathbf{\Lambda}^s \overline{\sigma} : \mathbf{\Lambda}^s \overline{\sigma}\right)^2}{|\mathbf{\Lambda}^s \overline{\sigma}|^2} \end{aligned} \tag{6}$$

Derivation of the lower bound

- From Thompson's Variational Principle we obtain

$$
\begin{aligned}
(C^e)^{-1}\mathbf{\Lambda}^s\overline{\sigma} : \mathbf{\Lambda}^s\overline{\sigma} &\leq (\theta_1(C^1)^{-1} + \theta_2(C^2)^{-1})\mathbf{\Lambda}^s\overline{\sigma} : \mathbf{\Lambda}^s\overline{\sigma} \\
&= \left(\frac{\theta_1}{2\mu_1} + \frac{\theta_2}{2\mu_2}\right)|\mathbf{\Lambda}^s\overline{\sigma}|^2
\end{aligned}
\tag{7}
$$

- Finally, from (6) and (7), after some simplification, we obtain

$$\langle \chi_1 \mathbf{\Lambda}^s \sigma(\mathbf{x}) : \sigma(\mathbf{x})\rangle \geq \theta_1 |\mathbf{\Lambda}^s\overline{\sigma}|^2$$

- Using Hölder's inequality we obtain

$$\langle \chi_i |\mathbf{\Lambda}^s \sigma(\mathbf{x})|^r \rangle^{1/r} \geq \theta_i^{1/r} |\mathbf{\Lambda}^s\overline{\sigma}|$$

Optimality

Have the bounds

$$\langle \chi_i |\mathbf{\Lambda}^s \sigma(\mathbf{x})|^r \rangle^{1/r} \geq \theta_i^{1/r} |\mathbf{\Lambda}^s \overline{\sigma}|$$

for $r \in [2, \infty]$, and

$$\| |\mathbf{\Lambda}^s \sigma(\mathbf{x})| \|_{L^\infty(Q)} \geq |\mathbf{\Lambda}^s \overline{\sigma}|$$
$$\| |\sigma(\mathbf{x})| \|_{L^\infty(Q)} \geq |\overline{\sigma}|$$

For $d = 2$, these lower bounds are attained by the stress field inside a rank-one laminate with layering direction $\mathbf{n} = \frac{1}{\sqrt{2}}(\mathbf{e^1} + \mathbf{e^2})$, where $\mathbf{e^1}$ and $\mathbf{e^2}$ are the eigenvectors of the macroscopic stress $\overline{\sigma}$.

Rank one layered material

Loading conditions

- Optimal lower bounds on the higher order moments and the L^∞ norm of the local stress and strain fields when the applied macroscopic loading is hydrostatic.
- Optimal lower bounds on the higher order moments and the L^∞ norm of the local stress and strain fields when the applied macroscopic loading is deviatoric.
- Optimal lower bounds on the higher order moments of the hydrostatic component of the local stress and strain fields for general applied macroscopic loading when the bulk moduli of the two materials are the same.
- Optimal lower bounds on the higher order moments and the L^∞ norm of the Von Mises equivalent stress and the deviatoric component of the strain for general applied macroscopic loading when the shear moduli of the two materials are the same.
- Optimal lower bounds on the higher order moments of the local stress and strain fields for a subspace of mixed mode loading characterized by a special dimensionless group of material parameters when the shear moduli of the two materials are the same.