August 5, 2008

Math Tune-Up Workshop, Calculus Day 1

The workshop participants should attempt to solve the problems by themselves, without the help of the assistants. Only when they are unable to solve the problem, should they receive help. The word problems are the most important since they require not only skills in doing calculations, but also good understanding of the material. If there is not enough time to solve all problems, the word problems should be discussed before the time runs out.

1. Describe and sketch the domain of the function
   (a) $f(x, y) = \sqrt{x + y}$
   (b) $f(x, y) = \sqrt{x} + \sqrt{y}$
   (c) $f(x, y) = \ln(9 - x^2 - 9y^2)$
   (d) $f(x, y, z) = \sqrt{4 - x^2 - y^2 - z^2}$
2. Describe and sketch the graph of the function
   (a) $f(x, y) = 1 - x - y$
   (b) $f(x, y) = x^2 + y^2$
   (c) $f(x, y) = x^2 - y^2$
   (d) $f(x, y) = \sqrt{x^2 + y^2}$
   (e) $f(x, y) = \sqrt{1 - x^2 - y^2}$
3. Draw the contour map of the function by showing several level curves
   (a) $f(x, y) = xy$
   (b) $f(x, y) = x^2 - y^2$
   (c) $f(x, y) = x - y^2$
4. Describe the level surfaces of the function
   (a) $f(x, y, z) = x + 3y + 5z$
   (b) $f(x, y, z) = x^2 + 3y^2 + 5z^2$
   (c) $f(x, y, z) = x^2 - y^2 + z^2$
   (d) $f(x, y, z) = x^2 + y^2$
5. Find all first and second partial derivatives
   (a) $f(x, y) = x^4 - 3x^2y^3$
   (b) $f(x, y) = x/(x + y)$
   (c) $f(x, y) = \sqrt{1 - x^2 - y^2}$
6. Verify that the function $u = 1/\sqrt{x^2 + y^2 + y^2}$ is a solution of the three-dimensional Laplace equation $u_{xx} + u_{yy} + u_{zz} = 0$.
7. Find an equation of the plane tangent to the graph of $z = \sqrt{4 - x^2 - 2y^2}$ at the point $(1, -1, 1)$.
8. Find the linearization of the function $f(x, y) = \arctan(x + 2y)$ at $(1, 0)$.

9. Use differentials to estimate the amount of tin in a closed cylindrical tin can with diameter 8 cm and height 12 cm if the tin is 0.04 cm thick.

10. If R is the total resistance of three resistors connected in parallel, with resistances R_1, R_2, and R_3, then

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}.$$

If the resistances are measured in ohms as $R_1 = 25\,\Omega$, $R_2 = 40\,\Omega$, and $R_3 = 50\,\Omega$ with a possible error of 0.5% in each case, use differentials to estimate the maximum error in the calculated value of R.

11. Find $\partial z/\partial s$ and $\partial z/\partial t$ when

    (a) $z = x^2 + xy + y^2$, $x = s + t$, $y = st$

    (b) $z = e^y \cos\theta$, $y = st$, $\theta = \sqrt{s^2 + t^2}$

12. The voltage V in simple electrical circuit is slowly decreasing as the battery wears out. The resistance R is slowly increasing as the resistor heats up. Use Ohm's Law, $V = IR$, to find out how the current I is changing at the moment when $R = 400\,\Omega$, $I = 0.08$ A, $dV/dt = -0.01$ V/s, and $dR/dt = 0.003$ Ω/s.

13. Find the gradient function of $f(x, y) = x \ln(x^2 + y^2)$.

14. Find the directional derivative of the function f at the given point in the direction $\mathbf{v}$

    (a) $f(x, y) = 1 + 2x\sqrt{y}$, $(3, 4)$, $\mathbf{v} = \langle 4, -3\rangle$

    (b) $f(x, y, z) = (x + 2y + 3z)^{3/2}$, $(1, 1, 2)$, $\mathbf{v} = 2\mathbf{j} - \mathbf{k}$

15. Find the equations of the tangent plane and the normal line to the surface at the given point

    (a) $x^2 - 2y^2 + z^2 + yz = 2$, $(2, 1, -1)$

    (b) $yz = \ln(x + z)$, $(0, 0, 1)$

16. The plane $y + z = 3$ intersects the cylinder $x^2 + y^2 = 5$ in an ellipse. Find parametric equations for the tangent line to this ellipse at the point $(1, 2, 1)$. If available, use Mathematica to draw the cylinder, plane, and line in the same graph.

17. Find the points on the surface $z^2 = xy + 1$ that are the closest to the origin. Solve this problem twice, with the aid of the Lagrange Multipliers Method and without.

18. A cardboard box without a lid is to have volume of $32,000$ cm^3. Find the dimensions that minimize the amount of cardboard used. Again, solve this problem with and without Lagrange Multipliers.