August 8, 2008

Math Tune-Up Workshop, Calculus Set 3

The workshop participants should attempt to solve the problems by themselves, without the help of the assistants. Only when they are unable to solve the problem, should they receive help. The word problems are the most important since they require not only skills in doing calculations, but also good understanding of the material. If there is not enough time to solve all problems, the word problems should be discussed before the time runs out.

1. Find the gradient vector field of $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$. Plot the field using Mathematica, if available.

2. Compute $\int_C xy\,dx + (x - y)\,dy$ where C consists of line segments from $(0, 0)$ to $(2, 0)$ and from $(2, 0)$ to $(3, 2)$.

3. Use Mathematica to plot the vector field $\mathbf{F}(x, y) = (x - y)\mathbf{i} + xy\mathbf{j}$ and the curve C, which is the arc of the circle $x^2 + y^2 = 4$ traversed counter-clockwise from $(2, 0)$ to $(0, -2)$. Can you guess whether the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is positive or negative from the graph? Evaluate the integral to confirm your guess.

4. Find the work done by the force field $\mathbf{F}(x, y) = x^2\mathbf{i} + xy\mathbf{j}$ on a particle that moves once around the circle $x^2 + y^2 = 4$ oriented in the clockwise direction.

5. Find the mass, center of mass, and moments of inertia about the three coordinate axis of a wire of constant density k that is bent in the form of the helix $x = 2\sin t$, $y = 2\cos t$, $z = 3t$, $0 \le t \le 2\pi$.

6. Find a function f such that $\nabla f = y^2 \cos z\mathbf{i} + 2xy \cos z\mathbf{j} - xy^2 \sin z\mathbf{k}$. Then evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $C : \mathbf{r}(t) = t^2\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}$, $0 \le t \le \pi$.

7. Evaluate the following integral twice: first directly, then by using Green's Theorem. $\int_C xy\,dx + x^2y^3\,dy$, where C is the triangle with vertices $(0, 0)$, $(1, 0)$, and $(1, 2)$.

8. Use Green's Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x, y) = \langle e^x + x^2y, x^y - xy^2\rangle$ and C is the circle $x^2 + y^2 = 25$ oriented clockwise.

9. Parametrize the the surface that is the hyperbolic paraboloid $z = y^2 - x^2$ that lies between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$. Then use this paramterization to compute the area of the surface.

10. Compute the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$ when $\mathbf{F}(x, y, z) = x\mathbf{i} - z\mathbf{j} + y\mathbf{k}$ and S is the part of the sphere $x^2 + y^2 + z^2 = 4$ in the first octant, with orientation towards the origin.

11. Compute the center of mass of the lamina of constant density obtained from the sphere of radius a centered at the origin by discarding the portion below the xy-plane.

12. Use Stoke's Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x, y, z) = (x + y^2)\mathbf{i} + (y + z^2)\mathbf{j} + (z + x^2)\mathbf{k}$ and C is the triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ oriented counter-clockwise as seen from above.

13. Use the Divergence Theorem to evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$ when $\mathbf{F}(x, y, z) = 4x^3z\mathbf{i} + 4y^3z\mathbf{j} + 3z^4\mathbf{k}$ and S is the sphere with radius R and center at the origin.