Instructions. Answer each of the questions on your own paper, and be sure to show your work so that partial credit can be adequately assessed. Put your name on each page of your paper.

1. [**12** Points] Divide the following propositional formulas into groups satisfying the properties: (1) all propositional formulas in the same group are logically equivalent; (2) formulas in different groups are not equivalent.

   (a) $p \implies q$

   (b) $(\sim q) \implies (\sim p)$

   (c) $\sim (p \vee q)$

   (d) $p \vee (\sim q)$

   (e) $\sim (\sim p \wedge q)$

   (f) $(\sim p) \vee q$

   ▶ **Solution.** Group 1: {(a), (b), (f)}; Group 2: {(c)}; Group 3: {(d), (e)}. ◀

2. [**10** Points] Recall that the Boolean set $\mathbb{B} = \{0, 1\}$. Define a Boolean function $f : \mathbb{B}^3 \to \mathbb{B}$ on the three Boolean variables p, q, and r by means of the formula

$$f(p, q, r) = 1 \text{ if and only if an odd number of } p,\ q,\ r \text{ have value } 1.$$

   (a) Fill in the truth table for f:

p	q	r	f
0	0	0	**0**
0	0	1	**1**
0	1	0	**1**
0	1	1	**0**
1	0	0	**1**
1	0	1	**0**
1	1	0	**0**
1	1	1	**1**

   (b) Express f using only $\sim$, $\vee$, and $\wedge$.

   ▶ **Solution.**

$$\boxed{f = ((\sim p) \wedge (\sim q) \wedge r) \vee ((\sim p) \wedge q \wedge (\sim r)) \vee (p \wedge (\sim q) \wedge (\sim r)) \vee (p \wedge q \wedge r)}$$

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3. [**8** Points] Let $P(x, y)$ be the predicate: "Team x in Conference y has a winning record." Express each of the following using quantifiers and the predicate $P(x, y)$.

   (a) Every conference has at least one team with a winning record.

▶ **Solution.** $\boxed{\forall y \exists x\ P(x,\, y)}$ ◀

(b) There is a conference in which no team has a winning record.

▶ **Solution.** $\boxed{\exists y \forall x\ (\sim P(x,\, y))}$ ◀

4. Let $P(x)$ be the predicate "x is odd", and let $Q(x)$ be the predicate "x is twice an integer". Determine whether the following quantified statements are true (and, of course, explain your answer):

(a) $(\forall x \in \mathbb{Z})(P(x) \implies Q(x))$.

▶ **Solution. False.** In order for this to be a true statement, the implication must be a true implication for *all* choices of $x \in \mathbb{Z}$. But for $x = 1$, the implication $P(1) \implies Q(1)$ is false, since $P(1)$ is true, but $Q(1)$ is false. ◀

(b) $(\forall x \in \mathbb{Z})(P(x)) \implies (\forall x \in \mathbb{Z})(Q(x))$.

▶ **Solution. True.** The statement $p = (\forall x \in \mathbb{Z})(P(x))$ is false, since it is not true that every integer is odd, while the statement $q = (\forall x \in \mathbb{Z})(Q(x))$ is false, since it is not true that every integer is twice an integer. Thus, the implication $p \implies q$ has truth value **True**, since the truth value of an implication False $\implies$ False is True. ◀

5. [**12** Points] Rewrite each of the following statements so that negations are applied exclusively to predicates (that is, so that no negation precedes a quantifier or an expression involving logical connectives).

(a) $\sim \forall x \forall y P(x,\, y)$

▶ **Solution.** $\boxed{\exists x \exists y\ (\sim P(x,\, y))}$ ◀

(b) $\sim \forall y \exists x P(x,\, y)$

▶ **Solution.** $\boxed{\exists y \forall x\ (\sim P(x,\, y))}$ ◀

(c) $\sim \forall y \forall x (P(x,\, y) \vee Q(x,\, y))$

▶ **Solution.** $\boxed{\exists x \exists y\ ((\sim P(x,\, y)) \wedge (\sim Q(x,\, y)))}$ ◀

(d) $\sim (\exists x (\sim P(x)) \wedge \forall y Q(y))$

▶ **Solution.** $\boxed{(\forall x\ P(x)) \vee (\exists y\ \sim Q(y))}$ ◀

6. [**12** Points] If x is an integer, consider the statement: "If x is even, then $x^2 + 1$ is odd."

(a) Write the converse statement.

▶ **Solution.** "If $x^2 + 1$ is odd, then x is even." ◀

(b) Write the contrapositive statement.

▶ **Solution.** "If $x^2 + 1$ is even, then x is odd." ◀

(c) Which, if any, of the original statement, the converse, or the contrapositive are true? (No proof required.)

▶ **Solution.** All three are true statements. ◀

7. [**8** Points] Let $f : X \to \mathbb{R}$ be a real valued function with domain X. In calculus you learned the definition of "f is a bounded function." Namely,

   The function f is bounded if there exists a positive number $M \in \mathbb{R}$ such that $|f(x)| \leq M$ for all $x \in X$.

   Write the negation of this definition, that is the definition of "f is an unbounded function," without using words like "not" or "it is not true that".

8. [**20** Points] For $n \geq 1$, let $F(n)$ be defined by the summation

$$(*) \qquad F(n) = \sum_{j=1}^{n} \frac{1}{(2j-1)(2j+1)}.$$

(a) Compute $F(1)$, $F(2)$, and $F(3)$.

▶ **Solution.**

$$F(1) = \frac{1}{(2 \cdot 1 - 1) \cdot (2 \cdot 1 + 1)} = \frac{1}{3}.$$

$$F(2) = \frac{1}{(2 \cdot 1 - 1) \cdot (2 \cdot 1 + 1)} + \frac{1}{(2 \cdot 2 - 1) \cdot (2 \cdot 2 + 1)} = \frac{1}{3} + \frac{1}{15} = \frac{2}{5}.$$

$$F(3) = F(2) + \frac{1}{(2 \cdot 3 - 1) \cdot (2 \cdot 3 + 1)} = \frac{2}{5} + \frac{1}{35} = \frac{3}{7}.$$

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(b) Find the number A_n so that $F(n+1) = F(n) + A_n$. (In other words, what is the number that must be added to the sum $F(n)$ in order to get the sum $F(n+1)$.)

▶ **Solution.** A_n is just the $(n+1)^{th}$ term in the sum $(*)$. That is,

$$A_n = \frac{1}{(2(n+1) - 1)(2(n+1) + 1)} = \frac{1}{(2n+1)(2n+3)}.$$

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(c) Prove by induction that

$$F(n) = \frac{n}{2n+1}, \text{ for } n \geq 1.$$

As usual, your argument should be written in complete sentences, and you must make clear where you are using your induction hypothesis. You may (and probably should) find your answers to earlier parts of this exercise to be of use in your argument.

► **Solution.** *Proof.* For $n \in \mathbb{N}^+$ let $P(n)$ be the statement "$\sum_{j=1}^{n} \frac{1}{(2j-1)(2j+1)} = \frac{n}{2n+1}$". We will prove by induction that $P(n)$ is true for all $n \geq 1$.
Base Case: For $n = 1$ the statement $P(1)$ is

$$\sum_{j=1}^{1} \frac{1}{(2j-1)(2j+1)} = \frac{1}{2 \cdot 1 + 1}.$$

This is true since both the left and right hand sides are equal to 1/3, as verified in Part (a), so the case $P(1)$ has been proved.
Induction Hypothesis. Now assume that the statement $P(k)$ is true for some k. This means that we are assuming the equality

$$(*_k) \qquad \sum_{j=1}^{k} \frac{1}{(2j-1)(2j+1)} = \frac{k}{2k+1}.$$

We must show that this implies the truth of the statement $P(k+1)$, which is

$$(*_{k+1}) \qquad \sum_{j=1}^{k+1} \frac{1}{(2j-1)(2j+1)} = \frac{k+1}{2(k+1)+1}.$$

So assuming the truth of Equation $(*_k)$, and starting with the left hand side of Equation $(*_{k+1})$, we find

$$\begin{aligned}
\sum_{j=1}^{k+1} \frac{1}{(2j-1)(2j+1)} &= \sum_{j=1}^{k} \frac{1}{(2j-1)(2j+1)} + \frac{1}{(2(k+1)-1)(2(k+1)+1)} \\
&= \frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)} \quad \text{from the assumed truth of } (*_k) \\
&= \frac{k(2k+3)+1}{(2k+1)(2k+3)} \\
&= \frac{2k^2+3k+1}{(2k+1)(2k+3)} \\
&= \frac{(2k+1)(k+1)}{(2k+1)(2k+3)} = \frac{k+1}{2k+3} \\
&= \frac{k+1}{2(k+1)+1},
\end{aligned}$$

where the last five equalities were just algebraic manipulation.

Thus, we have shown that if k is an arbitrary natural number such that $P(k)$ is a true statement, then so is $P(k+1)$ a true statement. By the induction principle, we conclude that $P(n)$ is true for all $n \geq 1$. □

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9. [**10** Points] The following summation formulas were proved in class:

$$\text{(a)}\ \sum_{j=0}^{n} q^n = \frac{q^{n+1}-1}{q-1},\ q \neq 1 \qquad \text{(b)}\ \sum_{j=1}^{n} j = \frac{n(n+1)}{2}.$$

Assuming these results, (or any method you prefer), evaluate the following sums.

(a) $\sum_{i=2}^{n} 7 \cdot 3^i$

▶ **Solution.** Letting $i = j + 2$ we see that $i = 2 \implies j = 0$ and $i = n \implies j = n - 2$, so

$$\sum_{i=2}^{n} 7 \cdot 3^i = \sum_{j=0}^{n-2} 7 \cdot 3^{j+2} = 7 \cdot 3^2 \sum_{j=0}^{n-2} 3^j = 63\frac{3^{n-1}-1}{2}.$$

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(b) $\sum_{k=1}^{n}(3k-1)$

▶ **Solution.**

$$\sum_{k=1}^{n}(3k-1) = 3\sum_{k=1}^{n} k - \sum_{k=1}^{n} 1 = 3\frac{n(n+1)}{2} - n = \frac{n(3n+1)}{2}.$$

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