Instructions. Answer each of the questions on your own paper, and be sure to show your work so that partial credit can be adequately assessed. Put your name on each page of your paper.

1. [**21 Points**] Let $A = \{a, b, c, d, e\}$, $B = \{1, a, b, 2, c\}$, and $C = \{3, c, a, 1, f\}$. Answer the following questions concerning these sets. Remember that the cardinality of a set means the number of elements in the set.

   (a) Write the set $A \cap B$.

   ▶ **Solution.** $A \cap B = \{a, b, c\}$ ◀

   (b) Write the set $A \cup B$.

   ▶ **Solution.** $A \cup B = \{1, 2, a, b, c, d, e\}$ ◀

   (c) Write the set $A \setminus (B \cap C)$.

   ▶ **Solution.** $A \setminus (B \cap C) = \{b, d, e\}$ ◀

   (d) Write the set $(A \cap C) \times (C \setminus B)$.

   ▶ **Solution.** $A \cap C = \{a, c\}$ and $C \setminus B = \{3, f\}$ so

   $$(A \cap C) \times (C \setminus B) = \{(a, 3), (a, f), (c, 3), (c, f)\}.$$

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   (e) What is the cardinality of $\mathcal{P}(A)$? Remember that $\mathcal{P}(A)$ denotes the power set of A.

   ▶ **Solution.** $|\mathcal{P}(A)| = 2^{|A|}$ and $|A| = 5$ so $|\mathcal{P}(A)| = 2^{|A|} = 2^5 = 32$ ◀

   (f) What is the cardinality of $A \times C$?

   ▶ **Solution.** $|A \times C| = |A| \cdot |C| = 5 \cdot 5 = 25$. ◀

   (g) Which set has more elements: $\mathcal{P}(A \times C)$ or $\mathcal{P}(A) \times \mathcal{P}(C)$?

   ▶ **Solution.** $|\mathcal{P}(A \times C)| = 2^{|A \times C|} = 2^{25}$ and $|\mathcal{P}(A) \times \mathcal{P}(C)| = |\mathcal{P}(A)| \cdot |\mathcal{P}(C)| = 2^5 \cdot 2^5 = 2^{10}$. Since $2^{25} > 2^{10}$, $\mathcal{P}(A \times C)$ has the most elements. ◀

2. [**6 Points**] If A, B, and C are arbitrary sets, draw a Venn diagram and shade the area corresponding the following set:

   $$[(A \cap B) \setminus C] \cup [(A \cap C) \setminus B].$$

▶ **Solution.** Done in class. ◀

3. **[12 Points]** How many subsets of the set $V = \{a, e, i, o, u, y\}$ contain exactly 4 elements? How many words of length 3 can be formed from the letters of V if no repetitions are allowed?

   ▶ **Solution.** The number of subsets of size k in an n element set is $\binom{n}{k}$. In this case $n = |V| = 6$ and $k = 4$ so there are $\binom{6}{4} = \frac{6!}{4!(6-2)!} = \frac{6\cdot 5}{2} = 15$ subset of V with 4 elements.

   A word of length 3 is just an ordered list of 3 elements of V. There are a total of $6 \cdot 5 \cdot 4 = 120$ such words. ◀

4. **[9 Points]** Which of the following are partitions of $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$? Explain your answers.

   (a) $\{\{1, 3, 5\}, \{1, 2, 6\}, \{4. 7, 8\}\}$
   (b) $\{\{1, 3, 5\}, \{2, 6, 7\}, \{4, 8\}\}$
   (c) $\{\{1, 5\}, \{2, 6\}, \{4, 8\}\}$

   ▶ **Solution.** Only (b) is a partition. In (a), the element 1 is repeated in two of the subsets, while in (c), the elements 3 and 7 are not present. In (b) each element of A appears in exactly one of the subsets, and this is what is meant by a partition. ◀

5. **[10 Points]** How many functions are there from a set X with 5 elements to a set Y with 7 elements? How many of these functions are injective?

   ▶ **Solution.** The number of functions from X to Y is $|Y|^{|X|} = 7^5 = 16807$. The number of injective functions is $7!/(7-2)! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 = 2520$. ◀

6. **[12 Points]** Let $f : A \to B$ be a function. Recall that the function f is said to be *surjective* (or *onto*) if for every $b \in B$ there is an $a \in A$ such that $f(a) = b$.

   (a) Give the definition of *injective* (or *one-to-one*) function by completing the sentence:

   The function f is injective if $f(a) = f(b)$ implies that $a = b$.

   Now suppose that $A = B = \mathbb{Z}$ = the set of integers, and f is given by the formula $f(x) = 4x + 7$. *The following two questions concern this specific function $f : \mathbb{Z} \to \mathbb{Z}$.*

   (b) Is f injective? Explain your answer using the definition above.

   ▶ **Solution.** f is injective since:

   $$f(x) = f(y) \implies 4x + 7 = 4y + 7 \implies 4x = 4y \implies x = y.$$

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(c) Is f surjective? Explain your answer using the definition above.

▶ **Solution.** f is not surjective since there is no integer x such that $f(x) = 0$ since this would imply that $4x + 7 = 0$. Since 4 does not divide -7 in $\mathbb{Z}$, this equation is not solvable. ◀

7. [**9 Points**] If the function $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = 3x - 2$, find the inverse function $f^{-1} : \mathbb{R} \to \mathbb{R}$.

▶ **Solution.** Solving $y = 3x - 2$ for x gives $x = (y + 2)/3$ so replace y by x on the right hand side to get $f^{-1}(x) = (x + 2)/3$. ◀

8. [**21 Points**] The following two permutations in $S(7)$ are given in two-rowed notation:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 4 & 5 & 6 & 7 & 2 & 1 \end{pmatrix} \text{ and } \tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 7 & 2 & 1 & 3 \end{pmatrix}.$$

(a) Compute the products $\sigma\tau$ and $\tau\sigma$. Express your answers in two-rowed notation. Are these two products equal?

▶ **Solution.**

$$\begin{aligned} \sigma\tau &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 7 & 2 & 1 & 4 & 3 & 5 \end{pmatrix} \\ \tau\sigma &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 7 & 2 & 1 & 3 & 5 & 4 \end{pmatrix} \end{aligned}$$

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(b) Compute σ^{-1}. Again express your answer in two-rowed notation.

▶ **Solution.** $\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 6 & 1 & 2 & 3 & 4 & 5 \end{pmatrix}$ ◀

(c) Write each of σ, τ, $\sigma\tau$ and $\tau\sigma$ as a product of disjoint cycles.

▶ **Solution.**

$$\begin{aligned} \sigma &= (1,\ 3,\ 5,\ 7)(2,\ 4,\ 6) \\ \tau &= (1,\ 4,\ 7,\ 3,\ 6)(2,\ 5) \\ \sigma\tau &= (1,\ 6,\ 3,\ 2,\ 7,\ 5,\ 4) \\ \tau\sigma &= (1,\ 6,\ 5,\ 3,\ 2,\ 7,\ 4) \end{aligned}$$

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