Exam 3 will be on Thursday, April 26, 2007. The syllabus will be Chapter SF together with pages 113–116 of Chapter EO. Following are some of the concepts and results you should know:

- Know basic set notation and various ways to combine sets.
- Know the algebra of set combinations (Theorem 1 Page 82).
- Know the definition of function and the basic properties of functions: surjective, injective, bijective, composition of functions, inverse of a function, criterion for when $f : X \to Y$ has an inverse f^{-1}, image and coimage of a function.
- Know how to represent functions between finite sets in two-rowed notation.
- The cardinality of X, denoted $|X|$, is the number of elements of X. Some formulas for the cardinality of combinations of sets X and Y:
  1. $|X \cup Y| = |X| + |Y| - |X \cap Y|$.
  2. $|X \times Y| = |X|\,|Y|$.
  3. $|\mathcal{P}(X)| = 2^{|X|}$ where $\mathcal{P}(X)$ denotes the power set of X, that is, $\mathcal{P}(X)$ is the set of all subsets of X.
  4. If $|X| = n$, then $|\mathcal{P}_k(X)| = \binom{n}{k}$ where $\mathcal{P}_k(X)$ denotes the set of all subsets of X of order k.
  5. $|\{\text{all functions } f : X \to Y\}| = |Y|^{|X|}$.
  6. Know the formula for the number of ordered lists of k elements from a set of n elements (Theorem 3, Page 89).
- Know what a relation on a set X is.
- Know the fundamental fact about an equivalence relation. Namely, an equivalence relation on set X determines a partition of X into disjoint sets called *equivalence classes*.
- $S(n)$ denotes the set of all permutations of the set $\{1, \ldots, n\}$ of integers from 1 to n. The cardinality of $S(n)$ is $|S(n)| = n!$.
- Know how to represent permutations in the two rowed notation, and how to multiply permutations using this notation.
- Know what a cycle of length r is. A cycle of length 2 is a *transposition*.
- Know what it means to say that two permutations π and σ are *disjoint*.
- Disjoint permutations commute.
- Know how to compute the cycle decomposition of permutations in $S(n)$.
- Know how to go back and forth between two rowed notation for permutations and cycle decompositions. Know how to multiply permutations given in either format and express the result in either two rowed or cycle notation.

Review Exercises

Be sure that you know how to do all assigned homework exercises. The following are a few supplemental exercises similar to those already assigned as homework. These exercises are listed randomly. That is, there is no attempt to give the exercises in the order of presentation of material in the text.

1. Evaluate each of the following sets, where

$$\begin{aligned} U &= \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} \\ A &= \{0, 1, 2, 3, 4, 5\} \\ B &= \{0, 2, 4, 6, 8, 10\} \\ C &= \{2, 3, 5, 7\}. \end{aligned}$$

   (a) $A \cup B$
   (b) $A \cap C$
   (c) $(\sim A) \cup B$
   (d) $A \cap B \cap C$
   (e) $A \cap (B \cup C)$
   (f) $(\sim A) \cap B \cap C$
   (g) $\sim (A \cup (\sim B))$
   (h) $A \setminus B$
   (i) $B \setminus A$
   (j) $A \setminus (B \setminus C)$
   (k) $C \setminus (B \setminus A)$

   ▶ **Solution.** Done in class. ◀

2. Describe two partitions of each of the following sets.

   (a) $\{x : x \text{ is an integer}\}$

   ▶ **Solution.** One partition is $\{E, O\}$ where E is the set of even integers and O is the set of odd integers. Another partition is given by congruence modulo 3: $\{3\mathbb{Z}, 3\mathbb{Z}+1, 3\mathbb{Z}+2\}$ ◀

   (b) $\{a, b, c, d\}$

   ▶ **Solution.** One partition: $\{\{a, b\}, \{c, d\}\}$;
   Second partition: $\{\{a, c\}, \{b, d\}\}$. ◀

   (c) $\{1, 5, 9, 11, 15\}$

   ▶ **Solution.** One partition: $\{\{1, 5\}, \{9, 11, 15\}\}$;
   Second partition: $\{\{1, 11\}, \{5, 9, 15\}\}$. ◀

3. Write out all of the partitions of the given set A.

   (a) $A = \{1, 2, 3\}$

► **Solution.** Done in class. (There were 5 partitions.) ◄

(b) $A = \{1, 2, 3, 4\}$

► **Solution.** There are a total of 15 partitions of the set with 4 elements:
$P_1 = \{1, 2, 3, 4\}$,
$P_2 = \{\{1\}, \{2, 3, 4\}\}$,
$P_3 = \{\{2\}, \{1, 3, 4\}\}$,
$P_4 = \{\{3\}, \{1, 2, 4\}\}$,
$P_5 = \{\{4\}, \{1, 2, 3\}\}$,
$P_6 = \{\{1, 2\}, \{3, 4\}\}$,
$P_7 = \{\{1, 3\}, \{2, 4\}\}$,
$P_8 = \{\{1, 4\}, \{2, 3\}\}$,
$P_9 = \{\{1\}, \{2, 3\}, \{4\}\}$,
$P_{10} = \{\{1, 2\}, \{3\}, \{4\}\}$,
$P_{11} = \{\{1, 3\}, \{2\}, \{4\}\}$,
$P_{12} = \{\{1, 4\}, \{2\}, \{3\}\}$,
$P_{13} = \{\{2, 4\}, \{1\}, \{3\}\}$,
$P_{14} = \{\{3, 4\}, \{1\}, \{2\}\}$,
$P_{15} = \{\{1\}, \{2\}, \{3\}, \{4\}\}$.
◄

4. (a) Give an example of a function $f : \mathbb{Z} \to \mathbb{Z}$ ($\mathbb{Z}$ is the set of integers) such that f is injective but not surjective.

► **Solution.** $f(n) = 2n$. ◄

(b) Give an example of a function $g : \mathbb{Z} \to \mathbb{Z}$ such that g is surjective but not injective.

► **Solution.** Define $f : \mathbb{Z} \to \mathbb{Z}$ by cases: $\begin{cases} x/2 & \text{if } x \text{ is even,} \\ x & \text{otherwise.} \end{cases}$
◄

(c) Give an example of a bijection $h : \mathbb{Z}^+ \to Y$ where $\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$ and $Y = \{2, 3, 4, \ldots\}$.

► **Solution.** $h(x) = x + 1$. ◄

5. How many functions are there from a set X with 4 elements to a set Y with 6 elements? How many of these functions are injective? How many are surjective?

► **Solution.** Number of functions: 6^4;

Number of injective functions: $6 \times 5 \times 4 \times 3 = 360$;

Number of surjective functions: 0. ◄

6. Let R be the relation on the integers $\mathbb{Z}$ defined by aRb if and only if $a^2 = b^2$. Verify that R is an equivalence relation on $\mathbb{Z}$. If $a \in \mathbb{Z}$, find the equivalence class $[a]_R$ of a. **Answer:** $[a]_R = \{\pm a\}$.

7. Let R be the relation on the set $A = \{1, 2, 3, 4, 5, 6\}$ defined by

$$R = \{(1,1),(1,3),(1,5),(2,2),(2,6),(3,1),(3,3),(3,5),(4,4),(5,1),(5,3),(5,5),(6,2),(6,6)\}.$$

Assuming that R is an equivalence relation, list all of the equivalence classes of R.

▶ **Solution.** $\{\{1, 3, 5\}, \{2, 6\}, \{4\}\}$ ◀

8. Assume that $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$ and $\tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$ are permutations in S_4. Compute each of the following elements of S_4:

(a) $\sigma\tau$ (b) $\tau\sigma$ (c) σ^2 (d) τ^2
(e) τ^3 (f) τ^4 (g) σ^{-1} (h) τ^{-1}

(a) **Answer:** $\sigma\tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$

(b) **Answer:** $\tau\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$

(c) **Answer:** $\sigma^2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$

(d) **Answer:** $\tau^2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$

(e) **Answer:** $\tau^3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$

(f) **Answer:** $\tau^4 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$

(g) **Answer:** $\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$

(h) **Answer:** $\tau^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$

9. Write each of the following permutations as a single cycle or a product of disjoint cycles.

(a) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 2 & 6 & 1 & 4 \end{pmatrix}$

(b) $(2\ \ 4)(4\ \ 5)(1\ \ 2)(2\ \ 5)$

(c) $(2\ \ 4\ \ 6)^{-1}(2\ \ 5)(3\ \ 2\ \ 1)$

(d) $(3\ \ 5\ \ 7)(1\ \ 3\ \ 5\ \ 6)(1\ \ 2\ \ 4)$

(a) **Answer:** (1, 3, 2, 5)(4, 6)

(b) **Answer:** (1, 4, 5)

(c) **Answer:** (1, 3, 5, 4, 6)

(d) **Answer:** (1, 2, 4, 5, 6)

10. Let α be a fixed element of $S(n)$ and define a function $\phi_\alpha : S(n) \to S(n)$ by the rule $\phi_\alpha(\sigma) = \alpha\sigma\alpha^{-1}$ for all $\sigma \in S(n)$.

(a) Show that ϕ_α is a bijective function.

► **Solution.** $\phi_\alpha(\sigma) = \phi_\alpha(\tau) \iff \alpha\sigma\alpha^{-1} = \alpha\tau\alpha^{-1} \iff \sigma = \tau$, so that ϕ_α is injective. Thus, the image of ϕ_α has the same number of elements as the codomain $S(n)$, namely $n!$. Since a proper subset of a finite set cannot have the same number of elements as the set itself, it follows that the image of ϕ_α is all of $S(n)$. That is, ϕ_α is surjective and hence bijective. ◄

(b) In $S(3)$, let $\alpha = (1, 2)$ and compute the function ϕ_α, that is, find $\phi_\alpha(\sigma)$ for all $\sigma \in S(3)$.

► **Solution.** The function $\phi_{(1,2)} : S(3) \to S(3)$ is given by the following table:

σ	$(\cdot)$	(1, 2)	(1, 3)	(2, 3)	(1, 2, 3)	(1, 3, 2)
$\phi_{(1,2)}(\sigma)$	$(\cdot)$	(1, 2)	(2, 3)	(1, 3)	(1, 3, 2)	(1, 2, 3)

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11. How many elements of $S(10)$ are products $(abcd)(efghi)$ of two disjoint cycles, one of length 4 and the other of length 5?

► **Solution.**

$$\boxed{\frac{1}{4}(10 \cdot 9 \cdot 8 \cdot 7) \times \frac{1}{5}(6 \cdot 5 \cdot 4 \cdot 3 \cdot 2) = 181,440}$$

◄

12. Write each of the following permutations as a product of disjoint cycles.

(a) (1, 2, 3)(1, 4, 5) $\boxed{(1, 4, 5, 2, 3)}$

(b) (1, 2, 3, 4)(1, 5, 6, 7) $\boxed{(1, 5, 6, 7, 2, 3, 4)}$

(c) (1, 2, 3, 4, 5)(1, 6, 7, 8, 9) $\boxed{(1, 6, 7, 8, 9, 2, 3, 4, 5)}$

(d) (1, 4)(2, 4)(3, 4)(1, 2)(2, 4)(2, 3) $\boxed{(1)(2)(3)(4) = (\cdot)}$