The Final Exam will be on Wednesday, May 9 from 5:30–7:30 P.M. in Lockett 134. The exam will be comprehensive, so you should collect all of the earlier homework assignments, exams and review sheets to use in preparing for the final exam. You can expect the problems to be like the problems on the three exams and homework problems. The following are a few problems that you can use to practice on. While these problems are typical of the level of difficulty that you can expect, you should not assume that every possible problem that could be on the exam is represented here.

Exercises

1. Write the truth table for the statement $(p \wedge q) \implies (p \vee q)$.

2. Verify that $(p \implies q) \vee (\sim q)$ is a tautology.

3. Let $P(x)$ and $Q(x)$ be predicates with domain set U. Recall that this means that x comes from the set U. Let A be the truth set of $P(x)$ and let B be the truth set of $Q(x)$. That is, $A = \{x \in U : P(x) \text{ is true}\}$ and $B = \{x \in U : Q(x) \text{ is true}\}$. In terms of A and B give:

   (a) The truth set of the predicate $P(x) \wedge Q(x)$.

   (b) The truth set of the predicate $P(x) \vee Q(x)$.

   (c) The truth set of the predicate $\sim P(x)$.

   (d) The truth set of the predicate $P(x) \wedge \sim Q(x)$.

   (e) The truth set of the predicate $P(x) \implies Q(x)$.

4. What is the negation of the statement $(\exists x P(x)) \implies (\forall x (P(x) \wedge Q(x)))$?

5. (a) Give a precise definition of the statement "The integer m is an odd integer."

   (b) Using your carefully presented definition of odd integer from part (a) prove that if m and n are two odd integers, then the product mn is also odd.

6. Let $A = \{1, 2, 3, 4, a, b\}$ and $B = \{3, b, c\}$.

   (a) List all of the subsets of B that contain exactly 2 elements.

   (b) What is the cardinality of the set $A \times B$?

   (c) Circle all of the following elements that are *not* in $A \times B$:

   $$(4, c), \quad (b, 3), \quad (b, 2), \quad (c, 4).$$

   (d) How many functions $f : A \to B$? How many functions $f : B \to A$?

7. Find the prime factorization of 660.

8. Let $a = 2730$ and $b = 570$.

   (a) Compute the greatest common divisor $d = \gcd(a, b)$.

   (b) Write d in the form $d = ra + sb$ for integers r and s.

   (c) Compute the least common multiple $m = \operatorname{lcm}(a, b)$.

9. Determine whether the following functions are **injective**, **surjective**, or both (**bijective**). **If bijective give the inverse function.**

(a) $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(n) = n + 1$.

(b) $f : \mathbb{N} \to \mathbb{N}$ defined by $f(n) = 3n + 5$. (Remember $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$.)

(c) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 5x - 2$.

(d) $f : \mathbb{R} \to \mathbb{Z}$ defined by $f(x) = [x]$ where $[x]$ is the largest integer n that is less than or equal to x.

10. Compute the following sum:
$$\sum_{i=0}^{n} \sum_{j=0}^{m} 4^i 7^j.$$

11. Prove by induction that $2^{2n} \equiv 1 \pmod{3}$ for all integers $n \geq 1$. Be sure to carefully identify each of the two parts of the induction argument and explain all of the steps involved in verifying each part.

12. Prove or disprove that $n^2 - 1$ is a composite integer whenever n is a positive integer greater than 1.

13. Prove or provide a counterexample: For all sets A, B, and C,
$$A \cap (B \cup C) = (A \cap B) \cup C.$$

14. (a) State the Chinese Remainder Theorem.

    (b) Find the two smallest positive solutions of the system of congruences
$$\begin{aligned} x &\equiv 3 \pmod{17} \\ x &\equiv 2 \pmod{35}. \end{aligned}$$

15. Suppose that $a = 2^{10} \cdot 3^7 \cdot 13^6 \cdot 29^8$ and $b = 2^9 \cdot 5^3 \cdot 11^4 \cdot 13^8$. Find the prime factorizations of (a, b), $[a, b]$, and ab.

16. Which of the integers 0, 1, 2, ..., 10 can be expressed as an integer linear combination of 12 and 20, that is, in the form $12s + 20t$ for integers s and t.

17. Prove each of the following statements by induction.

    (a) $2 \mid (n^2 + 3n)$ for every positive integer n.

    (b) For every positive integers n, there are integers r_n and s_n such that $5r_n + 6^n s_n = 1$.

18. Prove each of the following statements using just the definition of $a \mid b$.

    (a) If x, y, and z are integers such that $x \mid y$ and $y \mid z$, then $x^2 \mid z^2$.

    (b) If x, y, and z are integers such that $x \mid y$ and $y \mid z$, then $xy \mid z^2$.

    (c) If x, y, and z are integers such that $x \mid y$ and $y \mid z$, then $x \mid z$.

19. Solve the congruence $7x \equiv 1 \mod 23$.

20. Solve the simultaneous congruences $x \equiv 2 \mod 9$, $x \equiv 4 \mod 10$.

21. State the Fundamental Theorem of Arithmetic in grammatically correct and clear English.

22. (a) State the binomial theorem.
    (b) What is the coefficient of x^9 in the expansion of $(x+2)^{12}$.

23. Determine if each of the following statements is true (T) or false (F). If false, give a counterexample; if true, provide a reason. For this exercise, a, b and c denote arbitrary integers.
    (a) If $\gcd(a,b)=1$ and $\gcd(a,c)=1$, then $\gcd(b,c)=1$.
    (b) If $b \mid a^2+1$, then $b \mid a^4+1$.
    (c) If $b \mid a^2-1$, then $b \mid a^4-1$.
    (d) if $c \mid ab$, then $c \mid a$ or $c \mid b$.