The first midterm exam will be on Thursday, February 15, 2007. The syllabus consists of Section 1 of Unit BF, Unit Lo, and Section 1 of Unit IS from the text. The table of contents lists the basic terms covered in each section and you should be sure that you know the meaning of all of those terms. Moreover, you should be able to do all of the assigned problems, both suggested and those that were turned in.

The following are a few problems that are representative of the type and difficulty of problems that might appear on your exam. The problems are listed randomly, and are not necessarily in any order reminiscent of the order that topics were covered in class.

1. (a) Consider the sequence 1, −2, 4, −8, 16, −32, ... Letting $a_0 = 1$, $a_1 = -2$, etc. give a general formula for a_i.

   ▶ **Solution.** $\boxed{a_n = (-2)^n}$ ◀

   (b) Give a formula for the sum $\sum_{i=0}^{n} a_i$.

   ▶ **Solution.** Use the formula for the sum of a finite geometric series

$$\sum_{i=0}^{n} q^i = \frac{q^{n+1} - 1}{q - 1}, \quad \text{if } q \neq 1$$

   (which was proved in class) with $q = -2$ to get

$$\sum_{i=0}^{n} a_i = \sum_{i=0}^{n} (-2)^i = \boxed{\frac{(-2)^{n+1} - 1}{-3}}.$$

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2. Convert the propositional formula $p \implies (q \implies r)$ into an equivalent formula using only the connectives $\wedge$, $\vee$, and $\sim$.

   ▶ **Solution.** The statement forms $p \implies q$ and $(\sim p) \vee q$ are equivalent (see Page 32 (Lo-7), the sentence before Example 4). Applying this twice gives that $p \implies (q \implies r)$ is equivalent to $\boxed{(\sim p) \vee (\sim q) \vee r}$. ◀

3. Which of the following are logically equivalent to $p \implies q$? Give the label of all choices that apply. You need not show any work.

   (a) $\sim (q \implies p)$

   (b) $(\sim p) \implies (\sim q)$

   (c) $(\sim q) \implies (\sim p)$ **Answer:** $\boxed{\text{(c), (g)}}$

   (d) $p \wedge \sim q$

   (e) $p \vee \sim q$

(f) $(\sim p) \wedge q$

(g) $(\sim p) \vee q$.

4. For each formula in the left column, find a logically equivalent formula in the right column. It is possible that some items in the right column will be used more than once, and it is certain that some items in the right column will not be used at all.

c.	$(p \implies q) \vee (q \implies p)$	**a.**	p
e.	$(p \implies q) \wedge (q \implies p)$	**b.**	$\sim p$
c.	$p \implies (q \implies p)$	**c.**	$p \implies p$
f.	$\sim (q \implies \sim p)$	**d.**	$p \wedge \sim p$
g.	$\sim (\sim p \implies q)$	**e.**	$p \iff q$
		f.	$p \wedge q$
		g.	None of the above.

5. Let $P(x, y, z)$ be the predicate "$x = z - y$," where $x, y, z \in \mathbb{N}$. Determine whether each of the following statements are true:

(a) $P(2, 5, 3)$

▶ **Solution. False** since $2 \neq 3 - 5$. ◀

(b) $(P(2, 5, 3) \wedge P(0, 4, 4)) \implies P(3, 6, 3)$

▶ **Solution. True** since $p \implies q$ is true if p and q both have the truth value 0 (or F), and this is the situation here: $(P(2, 5, 3) \wedge P(0, 4, 4))$ is false since $P(2, 5, 3)$ is false and $P(3, 6, 3)$ is false since $3 \neq 3 - 6$. ◀

(c) $(\exists x \in \mathbb{N})(\forall y \in \mathbb{N})P(x, x, y)$

▶ **Solution. False.** There is no $x \in \mathbb{N}$ such that $x = y - x$ is true for *all* $y \in \mathbb{N}$. ◀

(d) $(\exists x \in \mathbb{N})(\forall y \in \mathbb{N})P(x, y, y)$

▶ **Solution. True.** If $x = 0$, then the equation $x = y - y$ is true for *all* $y \in \mathbb{N}$. ◀

(e) $(\forall x \in \mathbb{N})(\exists y \in \mathbb{N})P(x, x, y)$

▶ **Solution. True.** Given $x \in \mathbb{N}$ choose $y = 2x$. Then the equation $x = y - x$ is true. ◀

6. Determine which of the following are true statements. Recall that $\mathbb{R}$ denotes the real numbers, $\mathbb{R}^+$ denotes the positive real numbers, $\mathbb{Z}$ denotes the integers, and $\mathbb{Z}^+$ denotes the positive integers.

(a) $\forall x \in \mathbb{R}^+, \exists y \in \mathbb{Z}^+$ such that $x = y + 1$.

▶ **Solution. False.** The claim is that any positive real number is one more than some integer. As a concrete counter-example, let $x = 1.5$. Then there is no $y \in \mathbb{Z}$ such that $1.5 = y + 1$. ◀

(b) $\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}$ such that $x = y + 1$.

▶ **Solution. True.** Every integer is one more than some other integer. ◀

(c) $\exists x \in \mathbb{R}$ such that $\forall y \in \mathbb{R}$, $x = y + 1$.

▶ **Solution. False.** There is no real number which is one more than every other real number. As a concrete counterexample, for any $x \in \mathbb{R}$, the choice $y = x$ does not satisfy the equation $x = y + 1$. ◀

(d) $\forall v \in \mathbb{R}^+$, $\exists u \in \mathbb{R}^+$ such that $uv < v$.

▶ **Solution. True.** The choice $u = 1/2$ works, independent of the v that we start with. That is, $\frac{1}{2}v < v$ for every positive real v. ◀

(e) $\exists u \in \mathbb{R}^+$, such that $\forall v \in \mathbb{R}^+$, $uv < v$.

▶ **Solution. True.** As above, the choice $u = 1/2$ works. That is, $\frac{1}{2}v < v$ for every positive real v. ◀

(f) $\forall x \in \mathbb{Z}$ and $\forall y \in \mathbb{Z}$, $\exists z \in Z$ such that $z = x - y$.

▶ **Solution. True.** This is just a restatement of the fact that the integers are closed under the operation of subtraction. That is, the difference of any two integers is an integer. ◀

(g) $\forall x \in D$, $(P(x) \vee Q(x))$
always has the same truth value as
$(\forall x \in D, P(x)) \vee (\forall x \in D, Q(x))$.

▶ **Solution. False.** As a counterexample, let $D = \mathbb{R}$ be the set of real numbers, let $P(x)$ be the predicate "$x \geq 0$", and let $Q(x)$ be the predicate "$x \leq 0$". The first quantified statement is true because every real number x is either ≥ 0 or ≤ 0, but the second quantified statement is false, because it is not true that *all* real numbers x are ≥ 0 or *all* real numbers x are ≤ 0. Thus, the two quantified statements do not always have the same truth value. ◀

(h) $\forall x \in D$, $(P(x) \wedge Q(x))$
always has the same truth value as
$(\forall x \in D, P(x)) \wedge (\forall x \in D, Q(x))$.

▶ **Solution. True.** The first statement has truth value 1 (or T) precisely when both $P(x)$ and $Q(x)$ are simultaneously true for *all* $x \in D$. The second statement has truth value 1 (or T) precisely when $P(x)$ is true for every $x \in D$ and simultaneously $Q(x)$ is true for every $x \in D$. Thus, the two statements always have the same truth value: true is every element of D is in the truth set of both $P(x)$ and $Q(x)$; false otherwise. ◀

7. Express each of the following assertions (semi-)formally. You may use quantifiers such as "$\forall$ rational x", "$\exists$ a positive integer y", Boolean connections, and predicates such as "$x < y$", "$x \le y$", "$x = y$", "$x \ne y$", and functions like addition, subtraction, multiplication, and division.

   (a) There is a smallest positive integer.

   ▶ **Solution.** $\exists$ a positive integer x such that $\forall$ positive integers y, $x \le y$. ◀

   (b) There is no smallest positive rational number.

   ▶ **Solution.** $\forall$ positive rational numbers x, $\exists$ a positive rational y such that $y < x$. ◀

   (c) For any irrationals $x < y$, there is a rational z between x and y.

   ▶ **Solution.** $\forall$ irrationals x, $\forall$ irrationals y, if $x < y$, $\exists$ a rational z such that $x < z < y$. ◀

8. Define a sequence a_n as follows. Let $a_1 = 1$ and if $n \ge 1$ is a natural number, then $a_{n+1} = \frac{n^2}{n+1}a_n$. Prove that for all positive integers n,

$$a_n = \frac{(n-1)!}{n}.$$

Recall that $m! = m(m-1)\cdots 2\cdot 1$ is the product of all the natural numbers between 1 and m.

▶ **Solution.** *Proof.* For $n \in \mathbb{N}^+$ let $P(n)$ be the statement "$a_n = \frac{(n-1)!}{n}$". We will prove by induction that $P(n)$ is true for all $n \ge 1$.

Base Case: For $n = 1$ the statement $P(1)$ is

$$a_1 = \frac{(1-1)!}{1}.$$

This is true since both the left and right hand sides are equal to 1, so the case $P(1)$ has been proved.

Induction Hypothesis. Now assume that the statement $P(k)$ is true for some k. This means that we are assuming the equality

$$(*_k) \qquad a_k = \frac{(k-1)!}{k}.$$

We must show that this implies the truth of the statement $P(k+1)$, which is

$$(*_{k+1}) \qquad a_{k+1} = \frac{((k+1)-1)!}{k+1}.$$

So assuming the truth of Equation $(*_k)$, and starting with the recursive definition of a_{k+1} in terms of a_k, we find

$$\begin{aligned} a_{k+1} &= \frac{k^2}{k+1} a_k \\ &= \frac{k^2}{k+1} \cdot \frac{(k-1)!}{k} \quad \text{from the assumed truth of } (*_k) \\ &= \frac{k(k-1)!}{k+1} = \frac{k!}{k+1} \\ &= \frac{(k+1)-1)!}{k+1}, \end{aligned}$$

where the last three equalities was just algebraic manipulation.

Thus, we have shown that if k is an arbitrary natural number such that $P(k)$ is a true statement, then so is $P(k+1)$ a true statement. By the induction principle, we conclude that $P(n)$ is true for all $n \geq 1$. □

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9. Consider the following (meaningless) statement: "If some ducks like reggae, then no cows wear neckties."

   (a) Write the converse: *If no cows wear neckties, then some ducks like reggae.*

   (b) Write the contrapositive: *If some cows wear neckties, then no ducks like reggae.*

10. Write each of the following sentences using propositional logic built up from these basic propositions:

    M = "Mary sings"

    B = "Beano cries"

    Z = "Zena sleeps"

    (a) If Mary doesn't sing, then Beano cries and Zena sleeps.

    ▶ **Solution.** $\boxed{(\sim M) \implies (B \wedge Z)}$ ◀

    (b) Mary sings only if Beano cries.

    ▶ **Solution.** $\boxed{M \implies B}$ ◀

    (c) For Zena to sleep, it is necessary that Mary sing.

    ▶ **Solution.** $\boxed{Z \implies M}$ ◀

(d) Either Beano cries or Zena sleeps, but not both.

▶ **Solution.** $\boxed{(B \vee Z) \wedge \sim (B \wedge Z)}$ ◀

11. Use mathematical induction to show that

$$\sum_{i=1}^{n} i(i!) = (n+1)! - 1.$$

▶ **Solution.** *Proof.* For $n \in \mathbb{N}^+$ let $P(n)$ be the statement "$\sum_{i=1}^{n} i(i!) = (n+1)! - 1$". We will prove by induction that $P(n)$ is true for all $n \geq 1$.

Base Case: For $n = 1$ the statement $P(1)$ is

$$1 \cdot (1!) = (1+1)! - 1.$$

This is true since both the left and right hand sides are equal to 1, so the case $P(1)$ has been proved.

Induction Hypothesis. Now assume that the statement $P(k)$ is true for some k. This means that we are assuming the equality

$$(*_k) \qquad \sum_{i=1}^{k} i(i!) = (k+1)! - 1.$$

We must show that this implies the truth of the statement $P(k+1)$, which is

$$(*_{k+1}) \qquad \sum_{i=1}^{k+1} i(i!) = ((k+1)+1)! - 1.$$

So assuming the truth of Equation $(*_k)$, and starting with the left hand side of Equation $(*_{k+1})$, we find

$$\begin{aligned}
\sum_{i=1}^{k+1} i(i!) &= \sum_{i=1}^{k} i(i!) + (k+1)((k+1)!) \\
&= ((k+1)! - 1) + (k+1)((k+1)!) \quad \text{from the assumed truth of } (*_k) \\
&= (k+1)!((k+1)+1) - 1 = ((k+1)+1)! - 1 \\
&= (k+2)! - 1,
\end{aligned}$$

where the last two equalities was just algebraic manipulation.

Thus, we have shown that if k is an arbitrary natural number such that $P(k)$ is a true statement, then so is $P(k+1)$ a true statement. By the induction principle, we conclude that $P(n)$ is true for all $n \geq 1$. □

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12. For each of the sums below, write it in summation notation. Then find and prove a formula for the sum in terms of n.

    Hint: You may choose to prove that your formulas are valid either by use of induction, or it could be easier to use an already proved formula such as $\sum_{i=1}^{n} i = n(n+1)/2$.

    (a) $3 + 7 + 11 + \cdots + (4n-1)$

    ▶ **Solution.**

$$\begin{aligned} 3 + 7 + 11 + \cdots + (4n-1) &= \sum_{k=1}^{n}(4k-1) \\ &= 4\sum_{k=1}^{n} k - \sum_{k=1}^{n} 1 \\ &= 4\frac{n(n+1)}{2} - n \\ &= 2n(n+1) - n = n(2n+1). \end{aligned}$$

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    (b) $1 + 5 + 9 + \cdots + (4n+1)$

    ▶ **Solution.**

$$\begin{aligned} 1 + 5 + 9 + \cdots + (4n+1) &= \sum_{k=0}^{n}(4k+1) \\ &= 4\sum_{k=0}^{n} k + \sum_{k=0}^{n} 1 \\ &= 4\sum_{k=1}^{n} k + (n+1) \\ &= 4\frac{n(n+1)}{2} + (n+1) \\ &= 2n(n+1) + (n+1) = (n+1)(2n+1). \end{aligned}$$

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    (c) $-1 + 2 - 3 + 4 - \cdots - (2n-1) + 2n$

    ▶ **Solution.** This can be written in summation notation as

$$\sum_{i=1}^{2n}(-1)^i i.$$

Checking this sum for the first few values of n suggests that the summation may have the value:

$$\sum_{i=1}^{2n}(-1)^i i = n, \quad \text{for } n \geq 1.$$

We will prove that this conjectured value is true for $n \geq 1$ by means of induction.

Proof. For $n \in \mathbb{N}^+$ let $P(n)$ be the statement "$\sum_{i=1}^{2n}(-1)^i i = n$". We will prove by induction that $P(n)$ is true for all $n \geq 1$.
Base Case: For $n = 1$ the statement $P(1)$ is

$$-1 + 2 \cdot 1 = 1.$$

This is true since both the left and right hand sides are equal to 1, so the case $P(1)$ has been proved.
Induction Hypothesis. Now assume that the statement $P(k)$ is true for some k. This means that we are assuming the equality

$$(*_k) \qquad \sum_{i=1}^{2k}(-1)^i i = k.$$

We must show that this implies the truth of the statement $P(k+1)$, which is

$$(*_{k+1}) \qquad \sum_{i=1}^{2(k+1)}(-1)^i i = k+1.$$

So assuming the truth of Equation $(*_k)$, and starting with the left hand side of Equation $(*_{k+1})$, we find

$$\begin{aligned}
\sum_{i=1}^{2(k+1)}(-1)^i i &= \sum_{i=1}^{2k}(-1)^i i + (-1)^{2k+1}(2k+1) + (2k+2) \\
&= k + (-1)^{2k+1}(2k+1) + (2k+2) \quad \text{from the assumed truth of } (*_k) \\
&= k - (2k+1) + (2k+2) \\
&= (k+1),
\end{aligned}$$

where the last two equalities was just algebraic manipulation.
Thus, we have shown that if k is an arbitrary natural number such that $P(k)$ is a true statement, then so is $P(k+1)$ a true statement. By the induction principle, we conclude that $P(n)$ is true for all $n \geq 1$. □

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Here is an alternate solution, based on the observation that we are asked to compute the difference between the sum of the first n even integers and the sum of the first n odd integers. Simply add these numbers by grouping in n pairs, that is adding

$$-\text{odd} + \text{next even}.$$

Each pair contributes 1 to the total sum:

$$\sum_{i=1}^{2n}(-1)^i i = \sum_{j=1}^{n}(-(2j-1)+2j) = \sum_{j=1}^{n} 1 = n.$$

13. Prove that $5^n + 5 < 5^{n+1}$ for all natural number $n \geq 1$.

► **Solution.** Again it is appropriate to prove this inequality by means of induction:

Proof. For $n \geq 1$ let $P(n)$ be the statement

$$\text{“}5^n + 5 < 5^{n+1}\text{”}.$$

We will prove by induction that $P(n)$ is true for all $n \geq 1$.

Base Case: For $n = 1$ the statement $P(1)$ is

$$5^1 + 5 < 5^{1+1}.$$

This is the inequality $6 < 25$ which is true, so the case $P(1)$ has been proved.

Induction Hypothesis. Now assume that the statement $P(k)$ is true for some k. This means that we are assuming the inequality

$$(*_k) \qquad 5^k + 5 < 5^{k+1}.$$

We must show that this implies the truth of the statement $P(k+1)$, which is

$$(*_{k+1}) \qquad 5^{k+1} + 5 < 5^{(k+1)+1}.$$

So assuming the truth of inequality $(*_k)$, and starting with the left hand side of inequality $(*_{k+1})$, we find

$$\begin{aligned} 5^{k+1} + 5 &= 5^k \cdot 5 + 5 \\ &= 5(5^k + 1) \\ &< 5(5^k + 5) \\ &< 5 \cdot 5^{k+1} \quad \text{from the assumed truth of } (*_k) \\ &= 5^{(k+1)+1} = 5^{k+2}, \end{aligned}$$

where the first three lines and last line were just algebraic manipulations.

Thus, we have shown that if k is an arbitrary natural number such that $P(k)$ is a true statement, then so is $P(k+1)$ a true statement. By the induction principle, we conclude that $P(n)$ is true for all $n \geq 1$. □

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