1. Find the truth tables of each of the following compound statements.

   (a) $(\sim (p \wedge q)) \wedge (p \vee \sim q)$,

   Answer:

p	q	$p \wedge q$	$\sim (p \wedge q)$	$\sim q$	$p \vee \sim q$	$(\sim (p \wedge q)) \wedge (p \vee \sim q)$
0	0	0	1	1	1	1
0	1	0	1	0	0	0
1	0	0	1	1	1	1
1	1	1	0	0	1	0

   (b) $[p \wedge (\sim p \vee q)] \vee [(\sim (p \wedge q)) \wedge (p \ \vee q)]$,

   Answer:

p	q	$\sim p$	$\sim p \vee q$	$p \wedge (\sim p \vee q)$	$p \wedge q$	$\sim (p \wedge q)$	$p \vee q$	$(\sim (p \wedge q)) \wedge (p \ \vee q)$	$[p \wedge (\sim p \vee q)] \vee [(\sim (p \wedge q)) \wedge (p \ \vee q)]$
0	0	1	1	0	0	1	0	0	0
0	1	1	1	0	0	1	1	1	1
1	0	0	0	0	0	1	1	1	1
1	1	0	1	1	1	0	1	0	1

   (c) $(\alpha \wedge \sim \beta) \vee (\beta \wedge \sim \gamma) \vee (\gamma \wedge \sim \alpha)$.

   Answer:

α	β	γ	$\sim \beta$	$\alpha \wedge \sim \beta$	$\sim \gamma$	$\beta \wedge \sim \gamma$	$\sim \alpha$	$\gamma \wedge \sim \alpha$	$(\alpha \wedge \sim \beta) \vee (\beta \wedge \sim \gamma)$	$(\alpha \wedge \sim \beta) \vee (\beta \wedge \sim \gamma) \vee (\gamma \wedge \sim \alpha)$
0	0	0	1	0	1	0	1	0	0	0
0	0	1	1	0	0	0	1	1	0	1
0	1	0	0	0	1	1	1	0	1	1
0	1	1	0	0	0	0	1	1	0	1
1	0	0	1	1	1	0	0	0	1	1
1	0	1	1	1	0	0	0	0	1	1
1	1	0	0	0	1	1	0	0	1	1
1	1	1	0	0	0	0	0	0	0	0

2. Which of the following Boolean functions are equal? Prove using truth tables.

   (a) $(p \wedge \sim q) \vee (\sim p \wedge q)$,

   (b) $(p \vee q) \wedge (\sim (p \wedge q))$,

   (c) $\sim ((p \wedge q) \vee ((\sim p) \wedge (\sim q)))$.

   ► **Solution.** The procedure is to compute the truth tables for each of the functions from parts (a), (b), and (c). The equality of the functions is then determined by comparing the rightmost columns of their truth tables.

   (a)

p	q	$\sim q$	$p \wedge \sim q$	$\sim p$	$\sim p \wedge q$	$(p \wedge \sim q) \vee (\sim p \wedge q)$
0	0	1	0	1	0	0
0	1	0	0	1	1	1
1	0	1	1	0	0	1
1	1	0	0	0	0	0

   (b)

p	q	$p \vee q$	$p \wedge q$	$\sim (p \wedge q)$	$(p \vee q) \wedge (\sim (p \wedge q))$
0	0	0	0	1	0
0	1	1	0	1	1
1	0	1	0	1	1
1	1	1	1	0	0

(c)

p	q	$p \wedge q$	$(\sim p) \wedge (\sim q)$	$(p \wedge q) \vee ((\sim p) \wedge (\sim q))$	$\sim ((p \wedge q) \vee ((\sim p) \wedge (\sim q)))$
0	0	0	1	1	0
0	1	0	0	0	1
1	0	0	0	0	1
1	1	1	0	1	0

Since all three of the functions have the same last column in the truth table, it follows that all three boolean functions are equal. ◀

3. Using the systematic approach illustrated in class and in the proof of Theorem 1, Page BF-5, express the Boolean functions given by the following truth tables using only $\sim$, $\vee$, and $\wedge$.

(a)

p	q	f
0	0	1
0	1	0
1	0	1
1	1	0

(b)

p	q	r	g
0	0	0	1
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	0

(c)

p	q	r	h
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	1

▶ **Solution.** The procedure is to use $\sim$ and $\wedge$ to get a basic function for each row that has a 1 in the last column. Then connect these functions by means of $\vee$. The results are:

(a) $((\sim p) \wedge (\sim q)) \vee ((\sim p) \wedge q)$

(b) $((\sim p) \wedge (\sim q) \wedge (\sim r)) \vee ((\sim p) \wedge q \wedge r) \vee (p \wedge q \wedge (\sim r))$

(c) $((\sim p) \wedge (\sim q) \wedge r) \vee (p \wedge (\sim q) \wedge (\sim r)) \vee (p \wedge q \wedge (\sim r)) \vee (p \wedge q \wedge r)$

◀

4. Classify each of the following statements as true, false, or not a valid mathematical statement. Explain your answer.

(a) An integer is a rational number. **True.**

(b) Let x be a real number. *Not a valid statement.*

(c) $4 = 2 + 2$ and $7 < \sqrt{50}$. **True**, since both parts of the "and" are true statements.

(d) 5 is an even integer and $16^{-1/4} = 1/2$. **False**, since "5 is an even integer" is a false statement.

(e) 5 is an even integer or $16^{-1/4} = 1/2$. **True**, since "$16^{-1/4} = 1/2$" is a true statement, and an "or" connector only requires one of the statements to be true.

(f) $4 \neq 2 + 2 \implies 7 < \sqrt{50}$. **True** since the right hand side of the implication is true.

(g) $4 = 2+2 \implies 7 > \sqrt{50}$. **False**, since a True statement implying a False statement, is a False implication.

5. Write down the negation of each of the following statements in clear and concise English. Do not use the expression "It is not the case that" in your answers.

(a) Either $a^2 > 0$ or a is not a real number.

▶ **Solution.** $a^2 \le 0$ and a is a real number. ◀

(b) Every integer is divisible by a prime.

▶ **Solution.** Some integer is not divisible by a prime. ◀

(c) For every real number x, there is an integer n such that $n > x$.

▶ **Solution.** There is a real number x such that $x \ge n$ for every integer n. ◀

(d) There exists a planar graph that cannot be colored with at most four colors.

▶ **Solution.** Every planar graph can be colored with at most four colors. ◀

(e) For all integers a and b, there exist integers q and r such that $b = qa + r$.

▶ **Solution.** There are integers a and b, such that $b \ne aq + r$ for all integers q and r. ◀

6. Write down the converse and the contrapositive of each of the following implications.

(a) If $\frac{a}{b}$ and $\frac{b}{c}$ are integers, then $\frac{a}{c}$ is an integer.

▶ **Solution. Converse:** If $\frac{a}{c}$ is an integer, then $\frac{a}{b}$ and $\frac{b}{c}$ are integers.
Contrapositive: If $\frac{a}{c}$ is not an integer, then $\frac{a}{b}$ is not an integer or $\frac{b}{c}$ is not an integer. ◀

(b) If $x^2 = x + 1$, then $x = (1+\sqrt{5})/2$ or $x = (1-\sqrt{5})/2$.

▶ **Solution. Converse:** If $x = (1+\sqrt{5})/2$ or $x = (1-\sqrt{5})/2$, then $x^2 = x+1$.
Contrapositive: If $x \ne (1+\sqrt{5})/2$ and $x \ne (1-\sqrt{5})/2$, then $x^2 \ne x+1$. ◀

(c) If n is an odd integer, then $n^2 + n - 2$ is an even integer.

▶ **Solution. Converse:** If n^2+n-2 is an even integer, then n is an odd integer.
Contrapositive: If $n^2 + n - 2$ is an odd integer, then n is an even integer. ◀

(d) If $p(x)$ is a polynomial of odd degree, then $p(x)$ has at least one real root.

▶ **Solution. Converse:** If the polynomial $p(x)$ has at least one real root, then $p(x)$ has odd degree.

Contrapositive: If the polynomial $p(x)$ has no real roots, then $p(x)$ has even degree.

◀

7. Which of the following is a tautology, a contradiction, or neither? (See page Lo-2 for the definitions of tautology and contradiction.)

   (a) $(p \wedge q) \vee (\sim (p \wedge q))$

   Answer: This is a **tautology** since either $(p \wedge q)$ or its negation $(\sim (p \wedge q))$ is true for any choices of true values for p and q.

   (b) $(p \wedge q \wedge r) \vee (p \wedge q \wedge (\sim r)) \vee (\sim (p \wedge q))$

   Answer: This is a **tautology**, which can be checked by computing the truth table, and finding that all entries in the last column are 1.

   (c) $(p \vee q \vee r) \wedge (p \vee (\sim r)) \wedge (\sim p) \wedge (\sim q)$.

   Answer: This is a **contradiction**, which can be checked by computing the truth table, and finding that all entries in the last column are 0.

8. Given:

   - Predicates $I(x) =$"x is wise" and $E(x, y) =$"x knows y", and a
   - domain $P = \{a, b, c\}$, and
   - the truth set of $I(x)$ is $\{a\}$, and
   - the truth set of $E(x, y)$ is $\{(a, a), (b, b), (c, c), (a, c), (a, b)\}$. [Remember that the truth set of a predicate is the set of elements in the domain that make the predicate a true statement.]

   Answer the following questions:

   (a) Is the statement $(\forall x \in P)(\exists y \in P) : I(y) \implies E(x, x)$ true?

   ▶ **Solution. True**, since $E(x, x)$ is true for all choices of x. ◀

   (b) Is the statement $(\exists y \in P) : (\forall x \in P) \sim E(x, y) \vee \sim I(x)$ true?

   ▶ **Solution. False**, since $I(a)$ is true so $\sim I(a)$ is false and $\sim E(a, y)$ is false for all choices of y since $E(a, y)$ is true for all $y \in P$. ◀

   (c) What is the opposite (in colloquial English) of "everyone who is wise knows themselves"?

▶ **Solution.** In predicate logic, using the given definitions of $I(x)$ and $E(x, y)$, the statement "everyone who is wise knows themselves" becomes:

$$(\forall x \in P), I(x) \implies E(x, x).$$

Using the rules of predicate logic, the negation is:

$$(\exists x \in P) :\sim (I(x) \implies E(x, x)),$$

or, using the negation rule for $\implies$, this becomes: $(\exists x \in P) : I(x) \wedge \sim E(x, x)$. In English, this becomes: Some wise person does not know him/herself. ◀

(d) Convert the following from predicate logic to colloquial English: $\exists x \in P \forall y \in P : E(x, y) \implies I(y)$.

▶ **Solution.** Somebody only knows wise people. ◀

(e) Convert the following from English to predicate logic: "anyone who knows everybody is wise."

▶ **Solution.** $\forall x \forall y E(x, y) \implies I(x)$ ◀