**Exercises for practice:**

From Section IS (Section 1) (page 157) in your text, do the following exercises:

1.1, 1.2, 1.3, 1.4, 1.6, 1.7, 1.8, 1.16.

These exercises have (as do all the exercises from the text) solutions in the Solutions section. I recommend that you try to do these exercises without first looking at the author's solution. Additionally it is recommended that you do the exercises completely, including writing up the solutions in your own words. When you try to write up a complete solution on your own, you will quickly find any gaps in your understanding.

**Exercises to turn in:**

1. Write each of the following sums without using the sum symbol $\sum$ and evaluate the sum.

   (a) $\sum_{i=1}^{5} i^2$

   ▶ **Solution.** $\boxed{\sum_{i=1}^{5} i^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 55}$ ◀

   (b) $\sum_{j=0}^{2} 3^{j+2}$

   ▶ **Solution.** $\boxed{\sum_{j=0}^{2} 3^{j+2} = 3^2 + 3^3 + 3^4 = 117}$ ◀

   (c) $\sum_{k=-1}^{4}(2k^2 - k + 1)$

   ▶ **Solution.**

$$\begin{aligned}\sum_{k=-1}^{4}(2k^2 - k + 1) &= (2(-1)^2 - (-1) + 1) + (2(0^2) - 0 + 1) \\ &\quad +(2(1^2) - 1 + 1) + (2(2^2) - 2 + 1) \\ &\quad +(2(3^2) - 3 + 1) + (2(4^2) - 4 + 1) \\ &= 59.\end{aligned}$$

◀

2. Prove the following by induction:

$$\sum_{j=1}^{n}(j+1)2^j = n2^{n+1}, \quad \text{for } n \geq 1.$$

*Proof.* For $n \in \mathbb{N}$ let $P(n)$ be the statement "$\sum_{j=1}^{n}(j+1)2^j = n2^{n+1}$". We will prove by induction that $P(n)$ is true for all $n \geq 1$.

*Base Case:* For $n = 1$ the statement $P(1)$ is

$$(1+1)2^1 = 1 \cdot 2^{1+1}.$$

This is true since both the left and right hand sides are equal to 4, so the case $P(1)$ has been proved.

*Induction Hypothesis.* Now assume that the statement $P(k)$ is true for some $k$. This means that we are assuming the equality

$$(*_k) \qquad \sum_{j=1}^{k}(j+1)2^j = k2^{k+1}.$$

We must show that this implies the truth of the statement $P(k+1)$, which is

$$(*_{k+1}) \qquad \sum_{j=1}^{k+1}(j+1)2^j = (k+1)2^{(k+1)+1}.$$

So assuming the truth of Equation $(*_k)$, and starting with the left hand side of Equation $(*_{k+1})$, we find

$$\begin{aligned} \sum_{j=1}^{k+1}(j+1)2^j &= \sum_{j=1}^{k}(j+1)2^j + (k+2)2^{k+1} \\ &= k2^{k+1} + (k+2)2^{k+1} \quad \text{from the assumed truth of } (*_k) \\ &= (2k+2)2^{k+1} = 2(k+1)2^{k+1} \\ &= (k+1)2^{k+2}, \end{aligned}$$

where the last two equalities was just algebraic manipulation.

Thus, we have shown that if $k$ is an arbitrary natural number such that $P(k)$ is a true statement, then so is $P(k+1)$ a true statement. By the induction principle, we conclude that $P(n)$ is true for all $n \geq 1$. □

3. Prove the following by induction:

$$\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{4}\right)\cdots\left(1 - \frac{1}{n}\right) = \frac{1}{n}, \quad \text{for } n \geq 2.$$

*Proof.* For $n \geq 2$ let $P(n)$ be the statement

$$\text{“}\prod_{j=2}^{n}\left(1 - \frac{1}{j}\right) = \frac{1}{n}\text{”}.$$

We will prove by induction that $P(n)$ is true for all $n \geq 2$.

*Base Case:* For $n = 2$ the statement $P(2)$ is

$$\left(1 - \frac{1}{2}\right) = \frac{1}{2}.$$

This is true since both the left and right hand sides are equal to $1/2$, so the case $P(2)$ has been proved.

*Induction Hypothesis.* Now assume that the statement $P(k)$ is true for some $k$. This means that we are assuming the equality

$$(*_k) \qquad \prod_{j=2}^{k}\left(1 - \frac{1}{j}\right) = \frac{1}{k}.$$

We must show that this implies the truth of the statement $P(k+1)$, which is

$$(*_{k+1}) \qquad \prod_{j=2}^{k+1}\left(1 - \frac{1}{j}\right) = \frac{1}{k+1}.$$

So assuming the truth of Equation $(*_k)$, and starting with the left hand side of Equation $(*_{k+1})$, we find

$$\begin{aligned}
\prod_{j=2}^{k+1}\left(1 - \frac{1}{j}\right) &= \left(\prod_{j=2}^{k}\left(1 - \frac{1}{j}\right)\right)\left(1 - \frac{1}{k+1}\right) \\
&= \frac{1}{k}\left(1 - \frac{1}{k+1}\right) \quad \text{from the assumed truth of } (*_k) \\
&= \frac{1}{k}\left(\frac{(k+1)-1}{k+1}\right) \\
&= \frac{1}{k+1},
\end{aligned}$$

where the last two equalities was just algebraic manipulation.

Thus, we have shown that if $k$ is an arbitrary natural number such that $P(k)$ is a true statement, then so is $P(k+1)$ a true statement. By the induction principle, we conclude that $P(n)$ is true for all $n \geq 2$. □

4. Prove the following inequality by induction:

$$\left(1 + \frac{1}{2}\right)^n \geq 1 + \frac{n}{2}, \quad \text{for } n \geq 1.$$

*Proof.* For $n \geq 1$ let $P(n)$ be the statement

$$\text{“}\left(1 + \frac{1}{2}\right)^n \geq 1 + \frac{n}{2}\text{”}.$$

We will prove by induction that $P(n)$ is true for all $n \geq 1$.

*Base Case:* For $n = 1$ the statement $P(1)$ is

$$\left(1 + \frac{1}{2}\right) \geq 1 + \frac{1}{2}.$$

This is true since both the left and right hand sides are equal to $3/2$, so the case $P(1)$ has been proved.

*Induction Hypothesis.* Now assume that the statement $P(k)$ is true for some $k$. This means that we are assuming the inequality

$$(*_k) \qquad \left(1 + \frac{1}{2}\right)^k \geq 1 + \frac{k}{2}.$$

We must show that this implies the truth of the statement $P(k + 1)$, which is

$$(*_{k+1}) \qquad \left(1 + \frac{1}{2}\right)^{k+1} \geq 1 + \frac{k + 1}{2}.$$

So assuming the truth of inequality $(*_k)$, and starting with the left hand side of inequality $(*_{k+1})$, we find

$$\begin{aligned} \left(1 + \frac{1}{2}\right)^{k+1} &= \left(1 + \frac{1}{2}\right)^k \left(1 + \frac{1}{2}\right) \\ &\geq \left(1 + \frac{k}{2}\right) \left(1 + \frac{1}{2}\right) \quad \text{from the assumed truth of } (*_k) \\ &= 1 + \frac{k + 1}{2} + \frac{k}{4} \\ &\geq 1 + \frac{k + 1}{2}, \end{aligned}$$

where the last two equalities was just algebraic manipulation and the fact that $k/4$ is positive.

Thus, we have shown that if $k$ is an arbitrary natural number such that $P(k)$ is a true statement, then so is $P(k + 1)$ a true statement. By the induction principle, we conclude that $P(n)$ is true for all $n \geq 1$. □