Exercises for practice:

From Section SF (Section 1) (pages 92 – 95) in your text, do the following exercises:
1.1, 1.3, 1.6, 1.12, 1.15, 1.21.

From Section SF (Section 2) (pages 105 – 108) in your text, do the following exercises:
2.1, 2.5, 2.10, 2.11, 2.13, 2.14, 2.15, 2.16

These exercises have (as do all the exercises from the text) solutions in the Solutions section. I recommend that you try to do these exercises without first looking at the author's solution. Additionally it is recommended that you do the exercises completely, including writing up the solutions in your own words. When you try to write up a complete solution on your own, you will quickly find any gaps in your understanding.

Exercises to turn in:

1. Determine the following sets, i.e., list their elements if they are nonempty, and write $\emptyset$ if they are empty.

   (a) $\{n \in \mathbb{N} : n^2 = 9\}$ **Answer:** $\{3\}$

   (b) $\{n \in \mathbb{Z} : n^2 = 9\}$ **Answer:** $\{\pm 3\}$

   (c) $\{x \in \mathbb{R} : x^2 = 9\}$ **Answer:** $\{\pm 3\}$

   (d) $\{n \in \mathbb{N} : 3 < n < 7\}$ **Answer:** $\{4, 5, 6\}$

   (e) $\{n \in \mathbb{Z} : 3 < |n| < 7\}$ **Answer:** $\{\pm 4, \pm 5, \pm 6\}$

   (f) $\{x \in \mathbb{R} : x^2 < 0\}$ **Answer:** $\emptyset$

2. Let $A = \{1, 2, 3\}$, $B = \{n \in \mathbb{N} : n \text{ is even}\}$, and $C = \{n \in \mathbb{N} : n \text{ is odd}\}$.

   (a) Determine $A \cap B$, $B \cap C$, $B \cup C$, $B \oplus C = (B \setminus C) \cup (C \setminus B)$. (See Page 82 for the definitions of these set operations, if you do not remember them.)

   ► **Solution.** $A \cap B = \{2\}$; $B \cap C = \emptyset$; $B \cup C = \mathbb{N}$; $B \oplus C = \mathbb{N}$. ◄

   (b) Compute the power set $\mathcal{P}(A)$.

   ► **Solution.** $\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, A\}$. ◄

   (c) Which of the following sets are infinite? $A \oplus B$, $A \oplus C$, $A \setminus C$, $C \setminus A$.

   ► **Solution.** $A \oplus B$, $A \oplus C$, and, $C \setminus A$ are infinite. ◄

3. Here are the rules defining two functions mapping $\mathbb{N}$ into $\mathbb{N}$: $f(n) = n + 1$ and $g(n) = \max\{0, n - 1\}$ for $n \in \mathbb{N}$.

   (a) Calculate $f(n)$ for $n = 0, 1, 2, 3, 4, 73$.

   ► **Solution.** $f(0) = 1$, $f(1) = 2$, $f(2) = 3$, $f(3) = 4$, $f(4) = 5$, $f(73) = 74$. ◄

   (b) Calculate $g(n)$ for $n = 0, 1, 2, 3, 4, 73$.

► **Solution.** $g(0) = 0$, $g(1) = 0$, $g(2) = 1$, $g(3) = 2$, $g(4) = 3$, $g(73) = 72$. ◄

(c) Show that f is injective but not surjective.

► **Solution.** $f(n) = f(m) \iff n + 1 = m + 1 \iff n = m$. Thus, whenever $f(n) = f(m)$ then $n = m$ so f is injective. Since $n + 1 \neq 0$ for any choice of $n \in \mathbb{N}$, it follows that 0 is not in the image of f, so f is not surjective. ◄

(d) Show that g is surjective but not injective.

► **Solution.** f is not injective since $g(0) = 0 = g(1)$ but $1 \neq 0$. If $n \in \mathbb{N}$ then so is $n + 1$. Since $n + 1 \neq 0$ for any $n \in \mathbb{N}$, it follows from the definition of g that $g(n + 1) = (n + 1) - 1 = n$. ◄

(e) Show that $g \circ f(n) = n$ for all $n \in \mathbb{N}$, but that $f \circ g(n) = n$ does not hold for all $n \in \mathbb{N}$.

► **Solution.** $f \circ g(0) = f(g(0)) = f(0) = 1 \neq 0$, but $g \circ f(n) = g(f(n)) = g(n + 1) = \max 0, (n + 1) - 1 = n$. ◄

4. Determine the inverse of each of the following functions from $\mathbb{R}$ to $\mathbb{R}$.

(a) $f(x) = 2x + 3$

► **Solution.** Solve $y = 2x + 3$ for x to get $x = (y - 3)/2$, and then replace y by x to give the expression for $f^{-1}(x)$. Thus, $f^{-1}(x) = (x - 3)/2$. ◄

(b) $g(x) = x^3 - 2$ **Answer:** $g^{-1}(x) = \sqrt[3]{(x + 2)}$.

(c) $h(x) = (x - 2)^3$ **Answer:** $h^{-1}(x) = \sqrt[3]{x} + 2$.

(d) $k(x) = \sqrt[3]{x} + 7$ **Answer:** $k^{-1}(x) = (x - 7)^3$.

5. How many functions are there from a set X with 3 elements to a set Y with 4 elements? How many of these functions are injective? How many are surjective?

► **Solution.** The number of functions is $|Y|^{|X|} = 4^3 = 64$. The number of injective functions is $4 \times 3 \times 2 = 24$ (4 ways to choose $f(1)$, 3 ways to choose $f(2)$ once $f(1)$ is chosen, and 2 ways to choose $f(3)$ once $f(1)$ and $f(2)$ are chosen. The product comes from the fact that these choices are independent of each other.) Since the image of f can be no larger than the domain X, there are no surjective functions from X to Y. ◄

6. Assume that $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}$ and $\tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}$ are permutations in S_4. Compute each of the following elements of S_4:

(a) $\sigma\tau$ (b) $\tau\sigma$ (c) σ^2 (d) τ^2
(e) τ^3 (f) τ^4 (g) σ^{-1} (h) τ^{-1}

(a) **Answer:** $\sigma\tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix}$

(b) **Answer:** $\tau\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix}$

(c) **Answer:** $\sigma^2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$

(d) **Answer:** $\tau^2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$

(e) **Answer:** $\tau^3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}$

(f) **Answer:** $\tau^4 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$

(g) **Answer:** $\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}$

(h) **Answer:** $\tau^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 4 \end{pmatrix}$

7. Write each of the following permutations as a single cycle or a product of disjoint cycles.

(a) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 6 & 4 & 5 & 3 & 2 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 4 \end{pmatrix}\begin{pmatrix} 1 & 5 \end{pmatrix}\begin{pmatrix} 1 & 2 \end{pmatrix}\begin{pmatrix} 3 & 5 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}^{-1}\begin{pmatrix} 2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$

(d) $\begin{pmatrix} 2 & 4 & 5 \end{pmatrix}\begin{pmatrix} 1 & 3 & 5 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$

(a) **Answer:** (2, 6)(3, 4, 5)

(b) **Answer:** (1, 2, 5, 3, 4)

(c) **Answer:** (1, 2)

(d) **Answer:** (1, 4)

Note: The two notations $(a,\, b,\, c)$ and $(a\ b\ c)$ are both legitimate notations for cycles. Either one can be used.