Instructions

- Answer each of the questions on your own paper, and be sure to show your work so that partial credit can be adequately assessed. Put your name on each page of your paper.
- You may use a scientific calculator, but it should not be needed. If you use a calculator you must still write out the operations done on the calculator to show that you know how to solve the problem.
- There are 6 problems, with a total of 100 points possible.

1. [**15 Points**] Let $f(x, y) = ye^{xy^2}$. Compute f_{xy}.

   ▶ **Solution.**

$$\begin{aligned} f_y &= e^{xy^2} + 2y^2xe^{xy^2} = (1 + 2y^2x)e^{xy^2} \\ f_{xy} &= (f_y)_x = 2y^2e^{xy^2} + (1 + 2y^2x)y^2e^{xy^2} \\ &= y^2e^{xy^2}(3 + 2y^2x). \end{aligned}$$

◀

2. [**15 Points**] Find the equation of the tangent plane to the surface $x^3y + z^2 = 3$ at the point $(-1, 1, 2)$.

   ▶ **Solution.** $f_x = 3x^2y$, $f_y = x^3$, and $f_z = 2z$, so at $(-1, 1, 2)$ these evaluate to $f_x = 3$, $f_y = -1$, $f_z = 4$. Thus, the equation of the tangent plane at this point is

$$3(x + 1) - (y - 1) + 4(z - 2) = 0$$

or, equivalently

$$3x - y + 4z = 4.$$

◀

3. [**15 Points**] Let $f(x, y) = \sqrt{x^2 + y^2}$.

   (a) Find the linear approximation of $f(x, y)$ at the point $(3, 4)$.

   ▶ **Solution.** Letting $z = f(x, y)$, the linear approximation at the point $(3, 4)$ is

$$f(x, y) \approx f(3, 4) + \frac{\partial f}{\partial x}(3, 4)\Delta x + \frac{\partial f}{\partial y}(3, 4)\Delta y.$$

   Evaluating gives $f(3, 4) = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$ and

$$\begin{aligned} \frac{\partial f}{\partial x}(3, 4) &= \left.\frac{x}{\sqrt{x^2 + y^2}}\right|_{(x, y)=(3, 4)} = \frac{3}{5} \\ \frac{\partial f}{\partial y}(3, 4) &= \left.\frac{y}{\sqrt{x^2 + y^2}}\right|_{(x, y)=(3, 4)} = \frac{4}{5}, \end{aligned}$$

so the linear approximation is

$$f(x,\, y) \approx 5 + \frac{3}{5}(x-3) + \frac{4}{5}(y-4).$$

◀

(b) Using your answer in part (a), approximate the number $\sqrt{(3.2)^2 + (3.9)^2}$.

▶ **Solution.** In this case $\Delta x = 3.2 - 3 = 0.2$ and $\Delta y = 3.9 - 4 = -0.1$. Thus, the linear approximation gives

$$\begin{aligned}\sqrt{(3.2)^2 + (3.9)^2} = f(3.2,\, 3.9) &\approx 5 + \frac{3}{5}\cdot(0.2) + \frac{4}{5}\cdot(-0.1)\\ &= 5 + \frac{0.6}{5} - \frac{0.4}{5} = 5 + \frac{0.2}{5}\\ &= 5 + \frac{1}{25} = 5.04.\end{aligned}$$

◀

4. **[20 Points]** Let $T(x,\, y,\, z) = 3x^2 + 2y^2 - 4z$.

(a) Find ∇T at $(-1,\, -3,\, 2)$.

▶ **Solution.** $\nabla T = 6x\mathbf{i} + 4y\mathbf{j} - 4\mathbf{k} = -6\mathbf{i} - 12\mathbf{j} - 4\mathbf{k}$ at $(-1,\, -3,\, 2)$. ◀

(b) Find the directional derivative of T at $(-1,\, -3,\, 2)$ in the direction of the vector $\mathbf{v} = -\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$.

▶ **Solution.** The unit direction vector is

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{-\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}}{\sqrt{(-1)^2 + (-2)^2 + 2^2}} = \frac{-\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}}{3}.$$

Thus,

$$D_{\mathbf{u}}T(-1,\, -3,\, 2) = (-6\mathbf{i} - 12\mathbf{j} - 4\mathbf{k}) \cdot \frac{-\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}}{3} = \frac{1}{3}(6 + 24 - 8) = \frac{22}{3}.$$

◀

(c) In which direction is T increasing most rapidly at $(-1,\, -3,\, 2)$, and what is the maximum rate of increase of T at this point?

▶ **Solution.** The most rapid increase is in the direction of the gradient. Thus, the unit direction vector is

$$\mathbf{u} = \frac{\nabla T}{|\nabla T|} = \frac{-6\mathbf{i} - 12\mathbf{j} - 4\mathbf{k}}{2\sqrt{9 + 36 + 4}} = \frac{-6\mathbf{i} - 12\mathbf{j} - 4\mathbf{k}}{14} = \frac{-3\mathbf{i} - 6\mathbf{j} - 2\mathbf{k}}{7}.$$

The maximum rate of increase of T at this point is $|\nabla T| = 14$. ◀

5. [**15 Points**] Let $f(x, y) = \dfrac{y}{x^2} + e^{xy}$, $x = r\cos t$, $y = r\sin t$. Compute $\dfrac{\partial f}{\partial t}$ when $r = 1$, $t = 0$.

▶ **Solution.** Use the chain rule.

$$\begin{aligned}\frac{\partial f}{\partial t} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} \\ &= \left(-\frac{2y}{x^3} + ye^{xy}\right)(-r\sin t) + \left(\frac{1}{x^2} + xe^{xy}\right)(r\cos t) \\ &= 0 + (1+1)\cdot 1 = 2.\end{aligned}$$

The last line is obtained by evaluating at $r = 1$, $t = 0$, so that $x = 1$, $y = 0$. ◀

6. [**20 Points**] Let $f(x, y) = x^3 + 3xy + y^3$.

   (a) Find all the critical points of $f(x, y)$

   ▶ **Solution.** Solve for the critical points:

$$\begin{aligned}\frac{\partial f}{\partial x} &= 3x^2 + 3y = 0 \\ \frac{\partial f}{\partial y} &= 3x + 3y^2 = 0.\end{aligned}$$

   From the first equation, $y = -x^2$ and substituting in the second equation gives $x + (-x^2)^2 = 0$. Thus $x + x^4 = 0 \implies x(1 + x^3) = 0 \implies x = 0$ or $x = -1$. If $x = 0$, then $y = 0$ and if $x = -1$, then $y = -1$. Thus, there are two critical points, $(0, 0)$ and $(-1, -1)$. ◀

   (b) Determine whether each critical point is a local minimum, local maximum, or saddle point.

   ▶ **Solution.** Use the second derivative test: Compute the second derivatives: $A = f_{xx} = 6x$, $B = f_{xy} = 3$, and $C = f_{yy} = 6y$.

   - Thus, at $(0, 0)$, the discriminant is $D = AC - B^2 = 0 \cdot 0 - 9 = -9 < 0$. Hence, $(0, 0)$ is a saddle point.
   - At $(-1, -1)$, the discriminant is $D = AC - B^2 = (-6)(-6) - 9 = 27 > 0$. Since $A = -6 < 0$, it follows that $(-1, -1)$ is a local maximum.

   ◀