Instructions

- Answer each of the questions on your own paper, and be sure to show your work so that partial credit can be adequately assessed. Put your name on each page of your paper.
- You may use a scientific calculator, but it should not be needed. If you use a calculator you must still write out the operations done on the calculator to show that you know how to solve the problem.
- There are 6 problems, with a total of 102 points possible.

1. [**17 Points**] Use Lagrange Multipliers to find the global *maximum and minimum* values of $f(x, y) = 2x + y$ subject to the constraint $x^2 + y^2 = 4$.

▶ **Solution.** Let $g(x, y) = x^2 + y^2$. Then solve the equations $\nabla f(x, y) = \lambda \nabla g(x, y)$, $g(x, y) = 4$ to find the critical points:

$$\begin{aligned} 2 &= \lambda 2x \\ 1 &= \lambda 2y \\ x^2 + y^2 &= 4. \end{aligned}$$

Divide the first equation by the second to get $x/y = 2$ so that $x = 2y$. Substitute this in the third equation $x^2 + y^2 = 4$ to get $4y^2 + y^2 = 4$ so $y = \pm\frac{2}{\sqrt{5}} = \pm\frac{2\sqrt{5}}{5}$. Then $x = 2y = \pm\frac{4\sqrt{5}}{5}$ Thus, there are two critical points: $P_1 = \left(\frac{4\sqrt{5}}{5}, \frac{2\sqrt{5}}{5}\right)$ and $P_2 = \left(-\frac{4\sqrt{5}}{5}, -\frac{2\sqrt{5}}{5}\right)$. Evaluating $f(x, y)$ at these points gives: $f(P_1) = 2\sqrt{5}$, $f(P_2) = -2\sqrt{5}$. Therefore, the global maximum is $2\sqrt{5}$, which occurs at P_1, and the global minimum is $-2\sqrt{5}$, which occurs at P_2. ◀

2. [**17 Points**] Compute $\displaystyle\int_0^1 \int_0^{1-y} (x - y^2)\, dx\, dy$.

▶ **Solution.**

$$\begin{aligned} \int_0^1 \int_0^{1-y} (x - y^2)\, dx\, dy &= \int_0^1 \left(\frac{x^2}{2} - xy^2\right)\Big|_{x=0}^{x=1-y} dy \\ &= \int_0^1 \left(\frac{(1-y)^2}{2} - (1-y)y^2\right) dy = \int_0^1 \left(\frac{1}{2} - y - \frac{y^2}{2} + y^3\right) dy \\ &= \left(\frac{y}{2} - \frac{y^2}{2} - \frac{y^3}{6} + \frac{y^4}{4}\right)\Big|_0^1 \\ &= \frac{1}{2} - \frac{1}{2} - \frac{1}{6} + \frac{1}{4} = \frac{1}{12}. \end{aligned}$$

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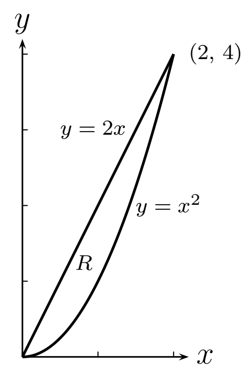
3. [**17 Points**]Let R be the finite region in the plane bounded by the two curves $y = x^2$ and $y = 2x$. Express the double integral $\iint_R f(x, y)\, dA$ as an iterated integral in two different ways.

▶ **Solution.** First draw the domain of integration R:



Then

$$\iint_R f(x, y)\, dA = \int_0^2 \int_{x^2}^{2x} f(x, y)\, dy\, dx$$
$$= \int_0^4 \int_{y/2}^{\sqrt{y}} f(x, y)\, dx,\, dy.$$

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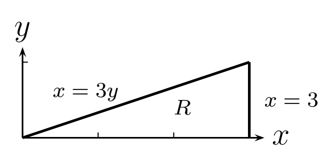
4. [**17 Points**]Evaluate the integral $\int_0^1 \int_{3y}^3 e^{x^2}\, dx\, dy$ by changing the order of integration.

▶ **Solution.** First draw the domain of integration R:



Then

$$\int_0^1 \int_{3y}^3 e^{x^2}\, dx\, dy = \int_0^3 \int_0^{x/3} e^{x^2}\, dy,\, dx$$
$$= \int_0^3 ye^{x^2}\Big|_0^{x/3}\, dx = \int_0^3 \frac{x}{3} e^{x^2}\, dx$$
$$= \frac{1}{6} e^{x^2}\Big|_0^3 = \frac{1}{6}(e^9 - 1).$$

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5. [**17 Points**] Evaluate $\iint_R e^{-(x^2+y^2)}\,dA$, where R is the region in the first quadrant of the x, y plane that lies between the circles with center at the origin and radii 2 and 4:

$$R = \left\{(x,\,y) : 4 \le x^2 + y^2 \le 16,\ x \ge 0, \text{and } y \ge 0\right\}.$$

Hint: Use polar coordinates.

▶ **Solution.** In polar coordinates, the region R is described by $0 \le \theta \le \pi/2$, $2 \le r \le 4$. Then

$$\begin{aligned}\iint_R e^{-(x^2+y^2)}\,dA &= \int_0^{\frac{\pi}{2}}\int_0^4 e^{-r^2}\,r\,dr\,d\theta \\ &= \int_0^{\frac{\pi}{2}} \left(-\frac{1}{2}e^{-r^2}\right)\Big|_{r=2}^{r=4} d\theta \\ &= \int_0^{\frac{\pi}{2}} \frac{1}{2}(e^{-4} - e^{-16})\,d\theta = \frac{\pi}{4}(e^{-4} - e^{-16}).\end{aligned}$$

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6. [**17 Points**] Compute the volume of the finite region Q bounded by the graphs of $z = x^2 + y^2$, $z = 0$, and $x^2 + y^2 = 9$ using cylindrical coordinates.

▶ **Solution.** The region Q can be described in polar coordinates as $0 \le \theta \le 2\pi$, $0 \le r \le 3$, $0 \le z \le r^2$. Thus the volume of Q is

$$\begin{aligned}\iiint_V dV &= \int_0^{2\pi}\int_0^3\int_0^{r^2} r\,dz\,dr\,d\theta \\ &= \int_0^{2\pi}\int_0^3 zr\big|_{z=0}^{z=r^2}\,dr\,d\theta = \int_0^{2\pi}\int_0^3 r^3\,dr\,d\theta \\ &= \int_0^{2\pi} \frac{r^4}{4}\Big|_0^3\,d\theta = \int_0^{2\pi}\frac{81}{4}\,d\theta = \frac{81\pi}{2}.\end{aligned}$$

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