Instructions

- Answer each of the questions on your own paper, and be sure to show your work so that partial credit can be adequately assessed. Put your name on each page of your paper.
- You may use a scientific calculator, but it should not be needed. If you use a calculator you must still write out the operations done on the calculator to show that you know how to solve the problem.
- There are 5 problems, with a total of 100 points possible. A 10 point bonus problem is on the second page.

1. [**20 Points**] Compute the line integral $\int_C (x+2y)\,dx + x^2\,dy$ where C is the line segment from (0, 1) to (2, 0).

   ▶ **Solution.** Parametrize C by $(1-t)(0,\,1) + t(2,\,0)$ for $0 \le t \le 1$ so that $x = 2t$, $y = 1-t$, $dx = 2dt$, $dy = -dt$. Then

$$\begin{aligned}\int_C (x+2y)\,dx + x^2\,dy &= \int_0^1 (2t + 2(1-t))2dt + 4t^2(-dt) \\ &= 4\int_0^1 (1-t^2)\,dt = 4t - \frac{4}{3}t^3\Big|_0^1 = \frac{8}{3}.\end{aligned}$$

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2. [**20 Points**] Let $\mathbf{F} = (2xy + z^3)\mathbf{i} + (x^2 + 2yz)\mathbf{j} + (y^2 + 3xz^2 - 1)\mathbf{k}$.

   (a) Verify that the vector field $\mathbf{F}$ is conservative.

   ▶ **Solution.** Writing $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ the necessary conditions for F to be conservative are:

$$\begin{aligned}P_y &= 2x = Q_x \\ P_z &= 3z^2 = R_x \\ Q_z &= 2y = R_y\end{aligned}$$

   Since these equalities are true, $\mathbf{F}$ is conservative. ◀

   (b) Using a systematic method, find a potential function $f(x,\,y,\,z)$ such that $\mathbf{F} = \nabla f$.

   ▶ **Solution.** Since $f_x = 2xy + z^3$, integrate with respect to x to get

$$f(x,\,y,\,z) = x^2y + xz^3 + g(y,\,z).$$

   Then

$$f_y = x^2 + g_y = x^2 + 2yz \implies g_y = 2yz.$$

Hence $g(y, z) = y^2 h(z)$, which gives

$$f(x,\, y,\, z) = x^2 y + xz^3 + y^2 z + h(z).$$

Then

$$f_z = 3xz^2 + y^2 + h'(z) = y^2 + 3xz^2 - 1 \implies h'(z) = -1 \implies h(z) = -z + K,$$

and thus,

$$f(x,\, y,\, z) = x^2 y + xz^3 + y^2 z - z + K,$$

where K is an arbitrary constant. ◀

(c) Compute $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is any curve from (1, 2, 0) to (0, 3, 4).

▶ **Solution.** By the fundamental theorem for line integrals,

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)) \\ &= f(0,\, 3,\, 4) - f(1,\, 2,\, 0) = 9 \cdot 4 - 4 + K - (2 + K) = 30. \end{aligned}$$

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3. [**20 Points**] Let C be the boundary of the rectangle with vertices (2, 1), (4, 1) (2, 4), (4, 4) and assume C is oriented counterclockwise. Use Green's theorem to evaluate the line integral

$$\int_C (y^2 + \cos x)\, dx + (x^2 - \sin y)\, dy.$$

▶ **Solution.** If D denotes the given rectangle $D = \{(x,\, y) : 2 \le x \le 4,\ 1 \le y \le 4\}$, then

$$\begin{aligned} \int_C (y^2 + \cos x)\, dx + (x^2 - \sin y)\, dy &= \iint_D \frac{\partial}{\partial x}(x^2 - \sin y) - \frac{\partial}{\partial y}(y^2 + \cos x)\, dA \\ &= \iint_D (2x - 2y)\, dA = \int_1^4 \int_2^4 (2x - 2y)\, dx\, dy \\ &= \int_1^4 (x^2 - 2xy)\Big|_2^4\, dy = \int_1^4 (16 - 8y - (4 - 4y))\, dy \\ &= \int_1^4 (12 - 4y)\, dy = (12y - 2y^2)\Big|_1^4 \\ &= (48 - 32) - (12 - 2) = 6. \end{aligned}$$

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4. [**20 Points**] Let $\mathbf{F} = (yz - xy)\mathbf{i} + (xz - xy)\mathbf{j} + (xz + yz)\mathbf{k}$.

(a) Calculate curl $\mathbf{F}$.

▶ **Solution.**

$$\begin{aligned}\operatorname{curl}\mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz - xy & xz - xy & xz + yz \end{vmatrix} \\ &= (z-x)\mathbf{i} - (z-y)\mathbf{j} + (z - y - (z-x))\mathbf{k} \\ &= (z-x)\mathbf{i} + (y-z)\mathbf{j} + (x-y)\mathbf{k}.\end{aligned}$$

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(b) Calculate div $\mathbf{F}$. div $\mathbf{F} = -y - x + (x - y) = 0$.

(c) Show that $\iint_S (\operatorname{curl}\mathbf{F})\cdot\mathbf{n}\,dS = 0$ for any surface S contained in the plane $x+y+z = 1$.

▶ **Solution.** For any surface in the plane $x + y + z = 1$, the unit normal vector $\mathbf{n}$ is the same as for the plane. Thus, $\mathbf{n} = \frac{\mathbf{i}+\mathbf{j}+\mathbf{k}}{\sqrt{3}}$. Then

$$\begin{aligned}\operatorname{curl}\mathbf{F}\cdot\mathbf{n} &= ((z-x)\mathbf{i} + (y-z)\mathbf{j} + (x-y)\mathbf{k}) \cdot \frac{\mathbf{i}+\mathbf{j}+\mathbf{k}}{\sqrt{3}} \\ &= \frac{1}{\sqrt{3}}((z-x) + (y-z) + (x-y)) = 0.\end{aligned}$$

Then

$$\iint_S \operatorname{curl}\mathbf{F}\cdot\mathbf{n}\,dS = \iint_S 0\,dS = 0.$$

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(d) Show that $\int_C \mathbf{F}\cdot d\mathbf{r} = 0$ for any simple closed curve in the plane $x + y + z = 1$.

▶ **Solution.** By Stokes' theorem, if S is the interior of the simple closed curve C, then

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \operatorname{curl}\mathbf{F}\cdot\mathbf{n}\,dS = 0,$$

by part (c).

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5. **[20 Points]** Let S be the surface of the solid bounded by the cylinder $x^2 + y^2 = 1$ and the planes $z = -1$ and $z = 2$, and let $\mathbf{F} = x^3\mathbf{i} + xe^z\mathbf{j} + 3zy^2\mathbf{k}$. Use the divergence theorem to evaluate the surface integral $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS$, where $\mathbf{n}$ is the outward unit normal.

▶ **Solution.** By the divergence theorem

$$\begin{aligned}\iint_S \mathbf{F}\cdot\mathbf{n}\,dS &= \iiint_E \operatorname{div}\mathbf{F}\,dV\\ &= \iiint_E (3x^2+3y^2)\,dV.\end{aligned}$$

The last integral is most conveniently evaluated by using cylindrical coordinates:

$$\begin{aligned}\iiint_E (3x^2+3y^2)\,dV &= \int_0^{2\pi}\int_0^1\int_{-1}^2 3r^2\cdot r\,dz\,dr\,d\theta\\ &= \int_0^{2\pi}\int_0^1 3\,r^3 z\Big|_{z=-1}^{z=2}\,dr\,d\theta = \int_0^{2\pi}\int_0^1 9r^3\,dr\,d\theta\\ &= \int_0^{2\pi}\frac{9}{4}\,d\theta = \frac{9\pi}{2}.\end{aligned}$$

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6. **Bonus Problem [10 Points]** The following are the main formulas of vector integral calculus, listed without hypotheses.

$$\int_a^b F'(x)\,dx = F(b) - F(a) \tag{1}$$

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)) \tag{2}$$

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\,dA \tag{3}$$

$$\int_C \mathbf{F} \cdot\, d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot \mathbf{n}\,dS \tag{4}$$

$$\iint_S \mathbf{F} \cdot \mathbf{n}\,dS = \iiint_E \operatorname{div} \mathbf{F}\,dV \tag{5}$$

In the following table match the equation number with the appropriate theorem.

Equation Number	Theorem
(3)	Green's Theorem
(1)	Fundamental Theorem of Calculus
(5)	Divergence Theorem
(2)	Fundamental Theorem for Line Integrals
(4)	Stokes' Theorem