Instructions

- Answer each of the questions on your own paper, and be sure to show your work so that partial credit can be adequately assessed. Put your name on each page of your paper.
- You may use a scientific calculator, but it should not be needed. If you use a calculator you must still write out the operations done on the calculator to show that you know how to solve the problem.
- There are 10 problems, with a total of 150 points possible.

1. [**15 Points**] Let $f(x, y, z) = xyz + z^3$ and let S be the surface given by the equation $f(x, y, z) = 12$.

   (a) Find the tangent plane to the surface S at the point $(-2, -1, 2)$.

   ▶ **Solution.** $\nabla f(x, y, z) = yz\mathbf{i} + xz\mathbf{j} + (xy + 3z^2)\mathbf{k}$ so $\nabla f(-2, -1, 2) = -2\mathbf{i} - 4\mathbf{j} + 14\mathbf{k}$ and the equation of the tangent plane at $(-2, -1, 2)$ is

   $$-2(x + 2) - 4(y + 1) + 14(z - 2) = 0$$

   which can also be written as

   $$-2x - 4y + 14z = 36 \quad \text{or} \quad x + 2y - 7z = -18.$$

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   (b) Compute the directional derivative $D_{\mathbf{u}}f(-2, -1, 2)$ if $\mathbf{u} = (\mathbf{i} + \mathbf{j})/\sqrt{2}$.

   ▶ **Solution.**

   $$\begin{aligned} D_{\mathbf{u}}f(-2, -1, 2) &= \nabla f(-2, -1, 2) \cdot \mathbf{u} \\ &= (-2\mathbf{i} - 4\mathbf{j} + 14\mathbf{k}) \cdot (\mathbf{i} + \mathbf{j})/\sqrt{2} \\ &= \frac{-6}{\sqrt{2}}. \end{aligned}$$

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   (c) Find a unit vector $\mathbf{v}$ such that $D_{\mathbf{v}}f(-2, -1, 2)$ is as large as possible.

   ▶ **Solution.** The direction of maximum increase is the gradient. Thus,

   $$\begin{aligned} \mathbf{v} &= \frac{\nabla f(-2, -1, 2)}{|\nabla f(-2, -1, 2)|} = \frac{-2\mathbf{i} - 4\mathbf{j} + 14\mathbf{k}}{|-2\mathbf{i} - 4\mathbf{j} + 14\mathbf{k}|} \\ &= \frac{-\mathbf{i} - 2\mathbf{j} + 7\mathbf{k}}{|-\mathbf{i} - 2\mathbf{j} + 7\mathbf{k}|} = \frac{-\mathbf{i} - 2\mathbf{j} + 7\mathbf{k}}{\sqrt{54}} \\ &= \frac{-\mathbf{i} - 2\mathbf{j} + 7\mathbf{k}}{3\sqrt{6}}. \end{aligned}$$

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2. **[10 Points]** Let $w(u, v) = u/v^2$, $u(x, y) = x + 2y$, $v(x, y) = x^2y$. Compute the partial derivatives w_x and w_y.

   ▶ **Solution.** Use the chain rule:

$$\begin{aligned} w_x &= w_u u_x + u_v v_x = \frac{1}{v^2} + \frac{-2u}{v^3} \cdot 2xy \\ w_y &= w_u u_y + w_v v_y = \frac{1}{v^2} \cdot 2 - \frac{2u}{v^3} \cdot x^2. \end{aligned}$$

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3. **[15 Points]** Find all of the critical points of the function

$$f(x, y) = 6x^2 - 3xy^2 + 3y^2 + 17$$

   and characterize each one as a local maximum, a local minimum, or a saddle point.

   ▶ **Solution.** To find the critical points, solve the equations

$$\begin{aligned} f_x &= 12x - 3y^2 = 0 \\ f_y &= -6xy + 6y = 0. \end{aligned}$$

   simultaneously for x and y. The second equation gives $y(x - 1) = 0$ so $y = 0$ or $x = 1$. If $y = 0$, the first equation gives $x = 0$, while if $x = 1$, the first equation gives $y^2 = 4$ so $y = \pm 2$. Thus, there are 3 critical points: $(0, 0)$, $(1, 2)$ and $(1, -2)$. To determine the type of critical point, compute the discriminant. First the second partial derivatives are $f_{xx} = 12$, $f_{yy} = -6x + 6$, $f_{xy} = -6y$. Then the discriminant D is

$$\begin{aligned} D &= f_{xx}f_{yy} - (f_{xy})^2 = 72(1 - x) - 36y^2 \\ &= 36(2(1 - x) - y^2). \end{aligned}$$

   Thus, for $(x, y) = (0, 0)$, $D = 72 > 0$ and $f_{xx} > 0$ so this critical point is a local minimum. If $(x, y) = (1, \pm 2)$, then $D = -144$ so that each of these two points are saddle points. ◀

4. **[15 Points]** Find the maximum and minimum values of the function

$$f(x, y) = 4x + 4y - x^2 - y^2$$

   subject to the condition

$$x^2 + y^2 = 2.$$

   ▶ **Solution.** Use Lagrange Multipliers. Let $g(x, y) = x^2 + y^2$. Then solve the simultaneous equations

$$\begin{aligned} \nabla f &= \lambda \nabla g \\ g(x, y) &= 2. \end{aligned}$$

These equations become

$$\begin{aligned} 4 - 2x &= \lambda 2x \\ 4 - 2y &= \lambda 2y \\ x^2 + y^2 &= 2. \end{aligned}$$

Multiply the first equation by x and the second equation by y to get

$$\begin{aligned} 4y - 2xy &= 2\lambda xy \\ 4x - 2xy &= 2\lambda xy. \end{aligned}$$

Subtract these equations to get $4y - 4x = 0$, which implies that $x = y$. Substituting into $x^2 + y^2 = 2$ gives $x^2 = 1$ so $x = \pm 1$. Since $x = y$ this gives two solutions to the equations: $(1, 1)$ and $(-1, -1)$. Since $f(1, 1) = 6$ and $f(-1, -1) = -10$, it follows that the maximum value of f on $x^2 + y^2 = 2$ is 6 and the minimum value is -10. ◀

5. [**20 Points**] Evaluate:

(a) $\displaystyle\int_0^1 \int_y^1 xy\,dx\,dy$

▶ **Solution.**

$$\begin{aligned} \int_0^1 \int_y^1 xy\,dx\,dy &= \frac{1}{2}\int_0^1 x^2y\Big|_{x=y}^{x=1}\,dy = \frac{1}{2}\int_0^1 (y - y^3)\,dy \\ &= \frac{1}{2}\left(\frac{y^2}{2} - \frac{y^4}{4}\right)\Bigg|_0^1 = \frac{1}{2}\left(\frac{1}{2} - \frac{1}{4}\right) = \frac{1}{8}. \end{aligned}$$

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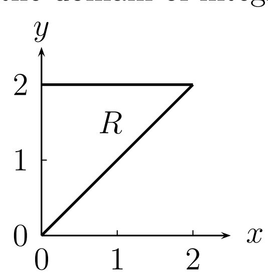
(b) $\displaystyle\int_0^2 \int_x^2 e^{y^2}\,dy\,dx$ (*Hint:* Sketch the domain and reverse the order of integration.)

▶ **Solution.** First draw the domain of integration R:



Then reverse the order of integration:

$$\begin{aligned} \int_0^2 \int_x^2 e^{y^2}\,dy\,dx &= \int_0^2 \int_0^y e^{y^2}\,dx\,dy = \int_0^2 xe^{y^2}\Big|_{x=0}^{x=y}\,dy \\ &= \int_0^1 ye^{y^2}\,dy = \frac{1}{2}e^{y^2}\Big|_0^2 = \frac{1}{2}(e^4 - 1). \end{aligned}$$

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6. **[15 Points]** Evaluate: $\displaystyle\int_0^1\int_0^x\int_0^y (y+xz)\,dz\,dy\,dx$

▶ **Solution.**

$$\begin{aligned}\int_0^1\int_0^x\int_0^y (y+xz)\,dz\,dy\,dx &= \int_0^1\int_0^x \left(yz + x\frac{z^2}{2}\right)\Bigg|_{z=0}^{z=y} dy\,dx \\ &= \int_0^1\int_0^x \left(y^2 + x\frac{y^2}{2}\right) dy\,dx = \int_0^1 \left(\frac{y^3}{3} + \frac{1}{6}xy^3\right)\Bigg|_{y=0}^{y=x} dx \\ &= \int_0^1 \left(\frac{x^2}{3} + \frac{x^4}{6}\right) dx = \left(\frac{x^4}{12} + \frac{x^5}{30}\right)\Bigg|_0^1 \\ &= \frac{1}{12} + \frac{1}{30} = \frac{7}{60}.\end{aligned}$$

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7. **[15 Points]** Find the volume of the region D lying above the plane $z=1$ and inside the sphere $x^2+y^2+z^2=4$.

▶ **Solution.** The plane $z=1$ intersects the sphere $x^2+y^2+z^2=4$ in the circle $x^2+y^2=3$. Thus, it is convenient to use cylindrical coordinates to calculate the volume. D is described in cylindrical coordinates by $0\le\theta\le 2\pi$, $0\le r\le\sqrt{3}$, $1\le z\le\sqrt{4-r^2}$, so

$$\begin{aligned}V = \iiint_D dV &= \int_0^{2\pi}\int_0^{\sqrt{3}}\int_1^{\sqrt{4-r^2}} r\,dz\,dr\,d\theta \\ &= \int_0^{2\pi}\int_0^{\sqrt{3}} rz\big|_1^{\sqrt{4-r^2}}\,dr\,d\theta \\ &= \int_0^{2\pi}\int_0^{\sqrt{3}} (r\sqrt{4-r^2} - r)\,dr\,d\theta \\ &= \int_0^{2\pi} \left(-\frac{1}{3}(4-r^2)^{3/2} - \frac{r^2}{2}\right)\Bigg|_0^{\sqrt{3}} d\theta \\ &= \int_0^{2\pi} \left(-\frac{1}{3} - \left(-\frac{8}{3}\right) - \frac{3}{2}\right) d\theta = 2\pi\left(\frac{7}{3} - \frac{3}{2}\right) = \frac{5\pi}{3}.\end{aligned}$$

Alternatively, the volume can be calculated by using spherical coordinates. ◀

8. **[15 Points]** Of the two vector fields

$$\mathbf{F}_1 = (2xy+z)\mathbf{i} + (x^2+\cos y)\mathbf{j} + x\mathbf{k} \quad \text{and} \quad \mathbf{F}_2 = 2xy\mathbf{i} + (x^2+\cos y)\mathbf{j} + x\mathbf{k},$$

one is conservative and the other is not.

(a) Determine which vector field is conservative, and write it in the form ∇f for some function f.

▶ **Solution.** Writing $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$, to check that $\mathbf{F}$ is conservative check that

$$\begin{aligned} R_y &= Q_z \\ R_x &= P_z \\ Q_x &= P_y. \end{aligned}$$

Since for $\mathbf{F}_1$, $R_y = 0 = Q_z$, $R_x = 1 = P_z$, and $Q_x = 2x = P_y$, it follows that $\mathbf{F}_1$ is conservative. To write $\mathbf{F}_1 = \nabla f$, $f_x = 2xy + z$ so $f(x, y, z) = x^2y + xz + g(y, z)$. Differentiating with respect to y gives $x^2 + g_y = f_y = x^2 + \cos y$. Thus, $g(y, z) = \sin y + h(z)$ and $f(x, y, z) = x^2y + xz + \sin y + h(z)$. Differentiating with respect to z gives $x + h'(z) = f_z = x$ so $h'(z) = 0$ and hence $h(z) = K$. Thus, $f(x, y, z) = x^2y + xz + \sin y + K$. ◀

(b) If $\mathbf{F}$ ($= \mathbf{F}_1$ or $\mathbf{F}_2$) is the conservative vector field, compute the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{r},$$

where C is the path consisting of the straight line segment from $(1, 0, 1)$ to $(0, \pi, 0)$ followed by a straight line segment from $(0, \pi, 0)$ to $(2, \pi, -1)$.

▶ **Solution.** Since $\mathbf{F}$ is conservative, the fundamental theorem for line integrals applies to give

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(2, \pi, -1) - f(1, 0, 1) = 4\pi - 2 - 1 = 4\pi - 3.$$

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9. **[15 Points]** Let C be the boundary of the triangle with vertices $(1, 1)$, $(2, 3)$ and $(2, 1)$, oriented positively (i.e. counterclockwise). Let $\mathbf{F}$ be the vector field

$$\mathbf{F}(x, y) = (e^x + y^2)\mathbf{i} + (xy + \cos y)\mathbf{j}.$$

Compute the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{r}.$$

▶ **Solution.** Use Green's Theorem. Let D be the given triangle. Then

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \iint_D (Q_x - P_y)\, dA = \iint_D (y - 2y)\, dA \\ &= \int_1^2 \int_1^{2x-1} -y\, dy\, dx = \int_1^2 -\frac{y^2}{2}\bigg|_{y=1}^{y=2x-1} dx \\ &= \int_1^2 \left(-\frac{(2x-1)^2}{2} + \frac{1}{2}\right) dx \\ &= \int_1^2 (-2x^2 + 2x)\, dx = \left(-\frac{2x^3}{3} + x^2\right)\bigg|_1^2 = -\frac{5}{3}. \end{aligned}$$

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10. [**15 Points**] Let S be the portion of the paraboloid $z = 1 - x^2 - y^2$ that lies on and above the plane $z = 0$. S is oriented by the normal directed upwards. If $\mathbf{F} = y\mathbf{i} - x\mathbf{j} + z\mathbf{k}$, then compute

$$\int_S \operatorname{curl} \mathbf{F} \cdot \mathbf{n}\, dS.$$

(*Hint:* Stokes Theorem)

► **Solution.** Use Stokes' Theorem. Let C be the boundary of the surface S oriented positively with respect to the upward pointing normal on S. Thus, C is the circle $x^2 + y^2 = 1$ in the xy plane oriented counterclockwise. By Stokes' theorem,

$$\int_S \operatorname{curl} \mathbf{F} \cdot \mathbf{n}\, dS = \int_C \mathbf{F} \cdot\, d\mathbf{r}.$$

Thus, we will compute the line integral. Use the parametrization $x = \cos t$, $y = \sin t$, $z = 0$ for $0 \le t \le 2\pi$, so that $dx = -\sin t\, dt$ and $dy = \cos t\, dt$, $dz = 0$. Hence,

$$\begin{aligned}\int_C \mathbf{F} \cdot\, d\mathbf{r} &= \int_C y\, dx - x\, dy + z\, dz \\ &= \int_0^{2\pi} ((\sin t)(-\sin t) - (\cos t)(\cos t))\, dt = -\int_0^{2\pi} dt = -2\pi.\end{aligned}$$

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