1. Let $\mathbf{F}(x, y) = xy\mathbf{i} + (y - 3x)\mathbf{j}$, and let C be the curve $\mathbf{r}(t) = t\mathbf{i} + (3t - t^2)\mathbf{j}$ for $0 \le t \le 2$. Compute $\int_C \mathbf{F} \cdot d\mathbf{r}$.

   ► **Solution.**

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt \\ &= \int_0^2 \langle t(3t - t^2),\, 3t - t^2 - 3t\rangle \cdot \langle 1,\, 3 - 2t\rangle\, dt \\ &= \int_0^2 t^3\, dt = \frac{1}{4}t^4 \Big|_0^2 = 4.\end{aligned}$$

◄

2. Evaluate the line integral $\int_C \frac{x^3}{y}\, ds$, where C is the portion of the graph $y = x^2/2$ for $0 \le x \le 2$.

   ► **Solution.** Note that this is the line integral of a scalar function with respect to arc length. We can parametrize C as $\mathbf{r}(t) = \langle t,\, \frac{1}{2}t^2\rangle$ for $0 \le t \le 2$. Then

$$\mathbf{r}'(t) = \langle 1,\, t\rangle \quad \text{and} \quad |\mathbf{r}'(t)| = \sqrt{1 + t^2},$$

   so that for $f(x, y) = x^3/y$:

$$\begin{aligned}\int_C \frac{x^3}{y}\, ds &= \int_a^b f(\mathbf{r}(t))\, |\mathbf{r}'(t)|\, dt = \int_0^2 \frac{t^3}{\frac{1}{2}t^2}\sqrt{1 + t^2}\, dt \\ &= \int_0^2 2t\sqrt{1 + t^2} \quad (\text{use substitution } u = 1 + t^2,\, du = 2t\, dt) \\ &= \int_0^5 \sqrt{u}\, du = \frac{2}{3}u^{3/2}\Big|_1^5 = \frac{2}{3}(t^{3/2} - 1).\end{aligned}$$

◄

3. Let $\mathbf{F}(x, y) = (ax^2y + y^3 + 1)\mathbf{i} + (2x^3 + bxy^2 + 2)\mathbf{j}$ be a vector field, where a and b are constants.

   (a) Find the values and a and b for which $\mathbf{F}$ is conservative.

   ► **Solution.** $P = ax^2y + y^3 + 1$ and $Q = 2x^3 + bxy^2 + 2$ so $P_y = ax^2 + 3y^2$ and $Q_x = 6x^2 + by^2$. Thus, $P_y = Q_x$ if and only if $a = 6$ and $b = 3$. ◄

   (b) For these values of a and b, find $f(x, y)$ such that $\mathbf{F} = \nabla f$.

▶ **Solution.** $f_x = 6x^2y + y^3 + 1 \implies f = 2x^3y + xy^3 + x + g(y)$. Therefore, $f_y = 2x^3 + 3xy^2 + g'(y)$. Setting this equal to Q gives $2x^3 + 3xy^2 + g'(y) = 2x^3 + 3xy^2 + 2$ so $g'(y) = 2$ and hence $g(y) = 2y + K$ where K is a constant. Hence,

$$f(x,\, y) = 2x^3y + xy^3 + x + 2y + K.$$

◀

(c) Still using the values of a and b from part (a), compute $\int_C \mathbf{F} \cdot d\mathbf{r}$ along the curve C such that $x = e^t \cos t$, $y = e^t \sin t$, $0 \le t \le \pi$.

▶ **Solution.** The curve C starts at $(1,\, 0)$ and ends at $(-e^\pi,\, 0)$, so by the fundamental theorem for line integrals (page 1075),

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(-e^\pi,\, 0) - f(1,\, 0) = -e^\pi - 1.$$

◀

4. Let $\mathbf{F}(x,\, y,\, z) = (\cos x + 2y^2 + 5yz)\mathbf{i} + (4xy + 5xz)\mathbf{j} + (5xy + 3z^2)\mathbf{k}$ on $\mathbb{R}^3$. Show that $\mathbf{F}$ is conservative, and find a potential function for $\mathbf{F}$.

▶ **Solution.** Writing $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$, the necessary and sufficient condition for $\mathbf{F}$ to be conservative is that $\mathrm{curl}(\mathbf{F}) = \mathbf{0}$. This means that $R_y = Q_z$, $R_x = P_z$, and $Q_x = P_y$. But

$$R_y = 5x = Q_z, \quad R_x = 5y = P_z, \quad \text{and} \quad Q_x = 4y + 5z = P_y.$$

Thus $\mathbf{F}$ is conservative. To find a potential function $f(x,\, y,\, z)$ such that $\nabla f = \mathbf{F}$, first set $f_x = P$ and integrate with respect to x to get

$$f(x,\, y,\, z) = \sin x + 2xy^2 + 5xyz + g(y,\, z).$$

Now differentiate with respect to y and set the result equal to Q to get $f_y = 4xy + 5xz + g_y = 4xy + 5xz$. Thus $g_y = 0$ so $g(y,\, z) = h(z)$ and $f(x,\, y,\, z) = \sin x + 2xy^2 + 5xyz + h(z)$. Now differentiate the result with respect to z and set the result equal to R to get $f_z = 5xy + h'(z) = 5xy + 3z^2$. Thus, $h'(z) = 3z^2$ so integrating gives $h(z) = z^3 + K$ where K is a constant. Thus,

$$f(x,\, y,\, z) = \sin x + 2xy^2 + 5xyz + z^3 + K.$$

◀

5. Verify that the vector field $\mathbf{F}(x,\, y) = (y^2 - y\cos x)\mathbf{i} + (2xy - \sin x + 1)\mathbf{j}$ is conservative. Then compute $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the curve $x = t^3 - t$, $y = 1 + t^2$, $1 \le t \le 2$.

▶ **Solution.** To verify that $\mathbf{F}$ is conservative, compute

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(y^2 - y\cos x) = 2y - \cos x, \quad \frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(2xy - \sin x + 1) = 2y - \cos x.$$

Since the two partial derivatives are equal, $\mathbf{F}$ is conservative. We can thus evaluate the line integral by the fundamental theorem for line integrals. We must first find a potential function f with $\nabla f = \mathbf{F}$. Thus, we must have

$$f_x = y^2 - y\cos x \quad \text{and} \quad f_y = 2xy - \sin x + 1.$$

Integrating f_x with respect to x gives $f(x, y) = xy^2 - y\sin x + g(y)$. Differentiate this with respect to y to get $f_y = 2xy - \sin x + g'(y)$. Setting this equal to the expression for f_y found earlier gives $g'(y) = 1$ so $g(y) = y + K$. Thus, our potential function is

$$f(x, y) = xy^2 - y\sin x + y + K.$$

The fundamental theorem for line integrals then gives

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_C \nabla f\cdot d\mathbf{r} = f(\mathbf{r}(2)) - f(\mathbf{r}(1)) = f(6, 5) - f(0, 2)\\ &= (6(25) - 5\sin(6) + 5 + K) - (0 - 0 + 2 + K) = 153 - 5\sin(6).\end{aligned}$$

◀

6. Let C be the circle $x^2 + y^2 = 100$, oriented counterclockwise. Find

$$\int_C (2xy + e^x)\, dx + (x^2 - \sin y + 3x)\, dy.$$

▶ **Solution.** Since C is a simple closed curve, Green's Theorem applies. First compute

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{\partial}{\partial x}(x^2 - \sin y + 3x) - \frac{\partial}{\partial y}(2xy + e^x) = (2x + 3) - (2x) = 3.$$

We can then calculate, using Green's Theorem (note that the inside of C is $D = \{x^2 + y^2 \leq 100\}$, the disc of radius 10 centered at the origin):

$$\begin{aligned}\int_C P\, dx + Q\, dy &= \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_D 3\, dA\\ &= 3(\text{Area}(D)) = 3\cdot\pi(10)^2 = 300\pi.\end{aligned}$$

◀

7. Let C be the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. Evaluate the line integral $\int_C \mathbf{F}\cdot d\mathbf{r}$ where $\mathbf{F}(x, y) = \dfrac{x}{a^2}\mathbf{i} + \dfrac{y}{b^2}\mathbf{j}$.

▶ **Solution.** Parametrize C by $x = a\cos t$, $y = b\sin t$, $0 \le t \le 2\pi$. Then

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_0^{2\pi}\left(\frac{a\cos t}{a^2}\mathbf{i} + \frac{b\sin t}{b^2}\mathbf{j}\right)\cdot(-a\sin t\mathbf{i} + b\cos t\mathbf{j})\,dt \\ &= \int_0^{2\pi}\left(\frac{-a^2\cos t\sin t}{a^2} + \frac{b^2\sin t\cos t}{b^2}\right)dt \\ &= \int_0^{2\pi} 3\,dt = 0.\end{aligned}$$

◀

8. Let C be the positively oriented closed curve formed by the parabola $y = x^2$ running from $(-1, 1)$ to $(1, 1)$ and by a straight line running back from $(1, 1)$ to $(-1, 1)$. Evaluate the line integral

$$\int_C y^2\,dx + 4xy\,dy$$

in two ways:

(a) directly, and

▶ **Solution.** Let $C = C_1 + C_2$, where C_1 is the curve running along the parabola and C_2 is the horizontal segment. Parametrize C_1 by $x(t) = t$, $y(t) = t^2$, $-1 \le t \le 1$. Then $dx = dt$ and $dy = 2t\,dt$, so the line integral over C_1 is

$$\int_{C_1} y^2\,dx + 4xy\,dy = \int_{-1}^{1}((t^2)^2 + 4tt^2(2t))\,dt = \int_{-1}^{1} 9t^4\,dt = \frac{9}{5}t^5\Big|_{-1}^{1} = \frac{18}{5}.$$

Parametrize C_2 by $x(t) = -t$, $y(t) = 1$, $-1 \le t \le 1$. Then $dx = -dt$ and $dy = 0$, so the line integral over C_2 is

$$\int_{C_2} y^2\,dx + 4xy\,dy = \int_{-1}^{1} -1\,dt = -t|_{-1}^{1} = -2.$$

Putting all this together we get

$$\int_C y^2\,dx + 4xy\,dy = \int_{C_1} y^2\,dx + 4xy\,dy + \int_{C_2} y^2\,dx + 4xy\,dy = \frac{18}{5} - 2 = \frac{8}{5}.$$

◀

(b) by using Green's theorem.

▶ **Solution.**

$$\int_C y^2\,dx + 4xy\,dy = \iint_R 2y\,dA = \int_{-1}^{1}\int_{x^2}^{1} 2y\,dy\,dx.$$

The inner integral is

$$\int_{x^2}^{1} 2y\,dy = y^2\Big|_{x^2}^{1} = 1 - x^4.$$

Thus, the outer integral is

$$\int_{-1}^{1} (1 - x^4)\,dx = \left(x - \frac{x^5}{5}\right)\Big|_{-1}^{1} = \frac{8}{5}.$$

◀

9. Let $\mathbf{F}(x,\, y,\, z) = y\mathbf{i} + z\mathbf{j} - 2x\mathbf{k}$ and let C be a simple closed curve in the plane $x + y + z = 1$. Show that $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$.

▶ **Solution.** Apply Stokes Theorem. Let S be the part of the plane $x + y + z = 1$ inside C. Then Stokes theorem states that

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\operatorname{curl} \mathbf{F}) \cdot \mathbf{n}\, dS,$$

where $\mathbf{n}$ is the unit normal vector to the surface S. Since S is the plane $x + y + z = 1$, the normal vector is $\mathbf{n} = \dfrac{\mathbf{i} + \mathbf{j} + \mathbf{k}}{\sqrt{3}}$. The curl of $\mathbf{F}$ is

$$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & -2x \end{vmatrix} = -\mathbf{i} + 2\mathbf{j} - \mathbf{k}.$$

Thus,

$$(\operatorname{curl} \mathbf{F}) \cdot \mathbf{n} = (-\mathbf{i} + 2\mathbf{j} - \mathbf{k}) \cdot \frac{\mathbf{i} + \mathbf{j} + \mathbf{k}}{\sqrt{3}} = 0$$

and hence

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\operatorname{curl} \mathbf{F}) \cdot \mathbf{n}\, dS = 0.$$

◀

10. Let Σ be the unit sphere centered at the origin, with the outward normal orientation, and let $\mathbf{F}(x,\, y,\, z) = x^3\mathbf{i} + y^3\mathbf{j} + z^3\mathbf{k}$. Compute the integral:

$$\iint_\Sigma \mathbf{F} \cdot \mathbf{n}\, dS.$$

Hint: Use the divergence theorem.

▶ **Solution.** Let B be the unit ball, so that Σ is the boundary of B. The divergence of $\mathbf{F}$ is $\operatorname{div} \mathbf{F} = 3x^2 + 3y^2 + 3z^2$. Then the divergence theorem gives

$$\iint_\Sigma \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_B \operatorname{div} \mathbf{F}\, dV = \iiint_B 3(x^2 + y^2 + z^2)\, dV.$$

This last integral can be evaluated in spherical coordinates ρ, ϕ, θ as

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{1} 3\rho^2\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \frac{12}{5}\pi.$$

◀

11. Let f be a scalar field (i.e., function) and $\mathbf{F}$ a vector field. Determine if each of the following expressions is meaningful. If it is meaningful, indicate whether it is a scalar field or a vector field.

Expression	Meaningful	Scalar	Vector
∇f	Yes		x
$\operatorname{div}\mathbf{F}$	Yes	x	
$\operatorname{curl} f$	No		
$\operatorname{curl}(\nabla f)$	Yes		x
$\nabla\mathbf{F}$	No		
$\operatorname{div}(\nabla\mathbf{F})$	Yes	x	