1. A function $u(x, y)$ is *harmonic* if it satisfies the partial differential equation $u_{xx}+u_{yy} = 0$. Determine if the function $u(x, y) = x^4 - 6x^2y^2 + y^4$ is harmonic.

   ▶ **Solution.** $u_x = 4x^3 - 12xy^2$, $u_{xx} = 12x^2 - 12y^2$, $u_y = -12x^2y + 4y^3$, $u_{yy} = -12x^2 + 12y^2$. Hence, $u_{xx} + u_{yy} = 12x^2 - 12y^2 - 12x^2 + 12y^2 = 0$, so u is harmonic. ◀

2. Let $w = u^2v + 2v^3 - u + 1$, where $u = (x - y^2)^3$ and $v = x^2 - y + 1$. Compute the partial derivatives w_x and w_y.

   ▶ **Solution.** Use the chain rule:

$$\begin{aligned} w_x &= w_u u_x + w_v v_x = (2uv - 1)3(x - y^2)^2 + (u^2 + 6v^2)2x \\ &= (2(x - y^2)^3(x^2 - y - 1) - 1)3(x - y^2)^2 + 2x((x - y^2)^6 + 6(x^2 - y + 1)^2) \\ w_y &= w_u u_y + w_v v_y = (2uv - 1)3(x - y^2)^2(-2y) + (u^2 + 6v^2)(-1) \\ &= (2(x - y^2)^3(x^2 - y - 1) - 1)3(x - y^2)^2(-2y) - ((x - y^2)^6 + 6(x^2 - y + 1)^2) \end{aligned}$$

   ◀

3. Find the rate of change of $f(x, y) = 3x^4 - xy + y^3$ at the point $P(1, 2)$ in the direction of the vector $\mathbf{a} = 2\mathbf{i} - \mathbf{j}$.

   ▶ **Solution.** A unit vector in the direction of $\mathbf{a}$ is

$$\mathbf{u} = \frac{\mathbf{a}}{|\mathbf{a}|} = \frac{2\mathbf{i} - \mathbf{j}}{\sqrt{5}}.$$

   Thus, the rate of change of f is the direction of $\mathbf{a}$ is the directional derivative $D_{\mathbf{u}}f(P) = \nabla f(P) \cdot \mathbf{u}$. Since $\nabla f = (12x^3 - y)\mathbf{i} + (-x + 3y^2)\mathbf{j}$, $\nabla f(P) = 10\mathbf{i} + 11\mathbf{j}$ so

$$D_{\mathbf{u}}f(P) = \nabla f(P) \cdot \mathbf{u} = (10\mathbf{i} + 11\mathbf{j}) \cdot \frac{2\mathbf{i} - \mathbf{j}}{\sqrt{5}} = \frac{9}{\sqrt{5}}.$$

   ◀

4. Let S be the surface defined by the equation

$$z = x^2y + xy^2 - 3y^2.$$

   (a) Find the equation of the tangent plane to S at the point $P = (2, 1, 3)$.

   ▶ **Solution.** Let $f(x, y) = x^2y + xy^2 - 3y^2$. Then

$$f_x = 2xy + y^2 \qquad \text{and} \qquad f_y = x^2 + y^2 - 6y.$$

   The equation of the tangent plane is

$$(z - 3) = f_x(x - 2) + f_y(y - 1) = 5(x - 2) + 2(y - 1).$$

   ◀

(b) Give a formula approximating the change Δz in z if x and y change by small amounts Δx and Δy.

▶ **Solution.**

$$\Delta z \approx 5\Delta x + 2\Delta y.$$

◀

(c) Approximate the value of z at the point (2.01, 1.01).

▶ **Solution.**

$$z = f(2.01,\, 1,01) = f(2,\, 1) + \Delta z \approx 3 + 7(.01) = 3.07.$$

◀

5. Let S be the surface

$$\frac{x^2}{1} + \frac{y^2}{4} + \frac{z^2}{9} = 1.$$

(a) Find a normal vector to the surface S at $\left(\frac{1}{\sqrt{3}},\, \frac{2}{\sqrt{3}},\, \sqrt{3}\right)$ on S.

▶ **Solution.** Let $f(x,\, y,\, z) = \frac{x^2}{1} + \frac{y^2}{4} + \frac{z^2}{9}$. Then S is the level surface $f(x,\, y,\, z) = 1$ and thus a normal vector at any point P is given by the gradient vector $\nabla f(P)$ at the point P. Thus, a normal vector to S at $P = \left(\frac{1}{\sqrt{3}},\, \frac{2}{\sqrt{3}},\, \sqrt{3}\right)$ is

$$\nabla f = \frac{2x}{1}\mathbf{i} + \frac{2y}{4}\mathbf{j} + \frac{2z}{9}\mathbf{k} = \frac{2}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{2}{3\sqrt{3}}\mathbf{k}.$$

◀

(b) At which point(s) on S is the vector $\langle 1,\, 1,\, 1\rangle$ normal to S?

▶ **Solution.** Set $\nabla f = k\langle 1,\, 1,\, 1\rangle$ to ensure that the gradient is parallel to $\langle 1,\, 1,\, 1\rangle$. Substitute $x = k/2$, $y = 2k$, $z = 9k/2$ into the equation $f(x,\, y,\, y) = 1$, i.e.,

$$\frac{k^2}{2^2} + \frac{(2k)^2}{4} + \frac{(9k/2)^2}{9} = 1$$

and solve for $k = \pm\frac{2}{\sqrt{14}}$. Thus, the two points are

$$(x,\, y,\, z) = \pm 1\sqrt{14}(1,\, 4,\, 9).$$

◀

6. Find *all* critical points of $f(x,\, y) = x^3 + y^4 - 6x - 2y^2$. Apply the second derivative test to each point and determine whether it is a local maximum, local minimum, or saddle point, or that the test fails.

► **Solution.** First compute the critical points. $f_x = 3x^2 - 6$ and $f_y = 4y^3 - 4y$, so setting $f_x = 0$ and $f_y = 0$ simultaneously, we find that $x = \pm\sqrt{2}$ and $y = 0, \pm 1$. Thus, there are a total of 6 critical points. To apply the second derivative test compute the discriminant $D = f_{xx}f_{yy} - (f_{xy})^2 = 24x(3y^2 - 1)$. The second derivative test then gives the following results: local maximum at $(-\sqrt{2}, 0)$, local minima at $(\sqrt{2}, 1)$ and $(\sqrt{2}, -1)$, and saddle points at $(\sqrt{2}, 0)$, $(-\sqrt{2}, 1)$ and $(-\sqrt{2}, -1)$. ◄

7. Use Lagrange multipliers to find the global maximum value and global minimum value of the function $f(x, y, z) = xyz$ subject to the constraint $g(x, y, z) = 1$ where $g(x, y, z) = \frac{x^2}{1} + \frac{y^2}{4} + \frac{z^2}{9}$.

   ► **Solution.** We set $\nabla f = \lambda \nabla g$, noting that since f achieves both positive and negative values on the ellipsoid $g(x, y, z) = 1$, we must have $xyz \neq 0$ at a global extreme point. Global extrema must exist since f is continuous on a closed bounded domain. The vector equation $\nabla f = \lambda \nabla g$ gives the simultaneous equations

$$
\begin{aligned}
yz &= \lambda 2x \\
xz &= \lambda \frac{2y}{4} \\
xy &= \lambda \frac{2z}{9}
\end{aligned}
$$

   By taking ratios of pairs of equations, we see that $y^2 = 4x^2$ and $z^2 = 9x^2$. Then we substitute for y and z in terms of x into the equation $g(x, y, z) = 1$ to get $3x^2 = 1$ so that $x = \pm\frac{1}{\sqrt{3}}$ and hence $y = \pm\frac{2}{\sqrt{3}}$ and $z = \pm\frac{3}{\sqrt{3}}$. Thus the global maximum and minimum values are $\pm\dfrac{2\sqrt{3}}{3}$ respectively. ◄

8. Evaluate:

   (a) $\displaystyle\int_0^1 \int_1^{e^y} \frac{y}{x}\, dx\, dy$

   ► **Solution.**

$$
\int_0^1 \int_1^{e^y} \frac{y}{x}\, dx\, dy = \int_0^1 y \ln x\big|_1^{e^y}\, dy = \int_0^1 y^2\, dy = \frac{1}{3}.
$$

◄

   (b) $\displaystyle\int_0^4 \int_{\sqrt{y}}^2 \sqrt{x^3+1}\, dx\, dy$. (*Hint:* Sketch the domain and reverse the order of integration.)

▶ **Solution.**

$$\int_0^4 \int_{\sqrt{y}}^2 \sqrt{x^3+1}\, dxdx = \int_0^2 \int_0^{x^2} \sqrt{x^3+1}\, dydx = \int_0^2 x^2\sqrt{x^3+1}\, dx$$
$$= \frac{1}{3}\int_1^9 \sqrt{u}\, du = 6 - \frac{2}{9} = \frac{52}{9}.$$

◀

9. Evaluate $\iiint_E z\, dV$ if E is the region defined by the inequalities $y \le z \le x$; $0 \le y \le x$; $0 \le x \le 1$.

▶ **Solution.**

$$\iiint_E z\, dV = \int_0^1 \int_0^x \int_y^x z\, dzdydx = \frac{1}{2}\int_0^1 \int_0^x (x^2 - y^2)\, dydx = \frac{1}{3}\int_0^1 x^3\, dx = \frac{1}{12}.$$

◀

10. Use spherical coordinates to evaluate $\iint_E x\, dV$ if E is the region defined by the inequalities $x^2 + y^2 + z^2 \le 1$; $x \ge 0$; $y \ge 0$; $z \ge 0$.

▶ **Solution.**

$$\iint_E x\, dV = \int_0^{\pi/2} \int_0^{\pi/2} \int_0^1 \rho \sin\phi \cos\theta \cdot \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$
$$= \frac{1}{4}\int_0^{\pi/2} \int_0^{\pi/2} \sin^2\phi \cos\theta\, d\theta\, d\phi = \frac{1}{4}\int_0^{\pi/4} \sin^2\phi\, d\phi$$
$$= \frac{1}{4}\int_0^{\pi/2} \frac{1-\cos 2\phi}{2}\, d\phi = \frac{\pi}{16}.$$

◀

11. Let C_1 be the line segment from (0, 0) to (1, 0), C_2 the arc of the unit circle running from (1, 0) to (0, 1) and let C_3 be the line segment from (0, 1) to (0, 0). Let C be the simple closed curve formed by C_1, C_2, and C_3, and let $\mathbf{F} = x^3\mathbf{i} + y^2x\mathbf{j}$. Calculate

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

in two ways:

(a) Directly,

▶ **Solution.** Parametrize C_1 by $x = t$, $y = 0$ for $0 \le t \le 1$ so that $dx = dt$ and $dy = 0$. Then

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} x^3\,dx + x^2y\,dy = \int_0^1 t^3\,dt = \left.\frac{t^4}{4}\right|_0^1 = \frac{1}{4}.$$

Parametrize C_2 by $x = \cos t$, $y = \sin t$ for $0 \le t \le \pi/2$, so that $dx = -\sin t\,dt$ and $dy = \cos t\,dt$. Then

$$\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_0^{\pi/2} (\cos^3 t(-\sin t) + \cos^2 t \sin t \cos t)\,dt = \int_0^{\pi/2} 0\,dt = 0.$$

Parametrize C_3 by $x = 0$, $y = 1 - t$ for $0 \le t \le 1$, so that $dx = 0$, $dy = -dt$. Then

$$\int_{C_3} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 0\,dt.$$

Therefore,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} + \int_{C_3} \mathbf{F} \cdot d\mathbf{r} = \frac{1}{4}.$$

◀

(b) using Green's theorem.

▶ **Solution.** Let R be the region bounded by C. Then $Q_x - P_y = 2xy$. Then Green's theorem gives

$$\int_C \cdot d\mathbf{r} = \iint_R (Q_x - P_y)\,dA = \int_0^{\pi/2} \int_0^1 2r^3 \cos\theta \sin\theta\,dr\,d\theta.$$

The inner integral is

$$\int_0^1 2r^3 \cos\theta \sin\theta\,dr = \left.\frac{r^4}{2} \cos\theta \sin\theta\right|_0^1 = \frac{1}{2}\cos\theta\sin\theta.$$

The outer integral is then

$$\int_0^{\pi/2} \frac{1}{2}\cos\theta\sin\theta\,dr = \left.\frac{1}{4}\sin^2\theta\right|_0^{\pi/2} = \frac{1}{4}.$$

◀

12. Let $\mathbf{F} = (6x + y + z)\mathbf{i} + (x - z + 7)\mathbf{j} + (x - y - 3z^2)\mathbf{k}$. Show that $\mathbf{F}$ is conservative and find a potential function for $\mathbf{F}$.

▶ **Solution.**

$$\operatorname{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 6x+y+z & x-z+7 & x-y-3z^2 \end{vmatrix}$$
$$= (-1-(-1))\mathbf{i} + (1-1)\mathbf{j} + (1-1)\mathbf{k}$$
$$= \mathbf{0}$$

so $\mathbf{F}$ is conservative. To find a potential function, we look for a function f such that

$$f_x = 6x+y+z$$
$$f_y = x-z+7$$
$$f_z = x-y+3z^2$$

simultaneously. Integrating the first equation with respect to x gives

$$f(x,\,y,\,z) = 3x^2 + xy + xz + g(y,\,z)$$

for some function $g(y,\,z)$. Differentiating with respect to y yields

$$f_y = x + g_y$$

and comparing with the second of the three equations gives $g_y = -z+7$, so integrating with respect to y gives

$$g(y,\,z) = -yz + 7y + h(z)$$

for some function $h(z)$. Substituting gives

$$f(x,\,y,\,z) = 3x^2 + xy + xz - yz + 7y + h(z).$$

Differentiating with respect to z gives $f_z = x-y+h_z$, and comparing with the third of the three equations gives $h_z = 3z^2$, so $h(z) = z^3+K$ for some constant K. Substituting gives

$$f(x,\,y,\,z) = 3x^2 + xy + xz - yz + 7y + z^3 + K.$$

◀

13. Let E be the 3-dimensional region described by the inequalities $x \geq 0$, $y \geq 0$, $x+y \leq 2$, and $0 \leq z \leq 3$. Let S be the entire boundary of E (all five faces)and let

$$\mathbf{F} = e^{-y^2}\mathbf{i} + \sin(e^x)\mathbf{j} + z^2\mathbf{k}.$$

Compute $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS$, where the normal $\mathbf{n}$ is oriented outward.

Hint: Use a theorem instead of trying to compute it directly.

▶ **Solution.** Apply the divergence theorem. First compute the divergence of $\mathbf{F}$:

$$\operatorname{div}\mathbf{F} = \frac{\partial}{\partial x}e^{-y^2} + \frac{\partial}{\partial y}\sin(e^x) + \frac{\partial}{\partial z}z^2 = 2z,$$

so the divergence theorem gives

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_E \operatorname{div}\mathbf{F}\,dV = \iiint_E 2z\,dV.$$

The cross section of E at height z is an isosceles right triangle R of side length 2 and area $\frac{1}{2}2^2 = 2$, so

$$\begin{aligned}\iiint_E 2z\,dV &= \int_{z=0}^{3}\iint_R 2z\,dA\,dz \\ &= \int_{z=0}^{3} 2z\text{Area}(R)\,dz \\ &= \int_{z=0}^{3} 4z\,dz \\ &= 2z^2\Big|_0^3 \\ &= 18.\end{aligned}$$

◀