Instructions

- Answer each of the questions on your own paper, and be sure to show your work so that partial credit can be adequately assessed. Put your name on each page of your paper.
- You may use a scientific calculator, but it should not be needed. If you use a calculator you must still write out the operations done on the calculator to show that you know how to solve the problem.
- There are 6 problems, with a total of 100 points possible.

1. [**15 Points**] Compute each of the following limits, if it exists, or else show why it does not exist:

   (a) $\displaystyle\lim_{(x,\,y)\to(0,\,0)} \frac{2x^2-y^2}{x^2+2y^2}$ (b) $\displaystyle\lim_{(x,\,y)\to(-4,\,3)} \frac{x+2y}{\sqrt{x^2+y^2}}$

   ▶ **Solution.** (a) Let $x = 0$. Then

   $$\lim_{(x,\,y)\to(0,\,0)} \frac{2x^2-y^2}{x^2+2y^2} = \lim_{y\to 0} \frac{-y^2}{2y^2} = \lim_{y\to 0} -\frac{1}{2} = -\frac{1}{2}.$$

   Now let $y = 0$. Then

   $$\lim_{(x,\,y)\to(0,\,0)} \frac{2x^2-y^2}{x^2+2y^2} = \lim_{x\to 0} \frac{2x^2}{x^2} = \lim_{y\to 0} 2 = 2.$$

   Since the limits are different when approaching $(0,\,0)$ along different paths, the limit $\displaystyle\lim_{(x,\,y)\to(0,\,0)} \frac{2x^2-y^2}{x^2+2y^2}$ does not exist.

   (b) Since $f(x,\,y) = \dfrac{x+2y}{\sqrt{x^2+y^2}}$ is a continuous function except for $(0,\,0)$, the limit can be calculated by evaluation:

   $$\lim_{(x,\,y)\to(-4,\,3)} \frac{x+2y}{\sqrt{x^2+y^2}} = \frac{-4+2\cdot 3}{\sqrt{(-4)^2+3^2}} = \frac{2}{5}.$$

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2. [**15 Points**] Let $f(x,\,y) = \ln(x^2+2y^2)$. Compute f_x, f_y, and f_{xy}.

   ▶ **Solution.**

   $$f_x = \frac{2x}{x^2+2y^2}$$
   $$f_y = \frac{4y}{x^2+2y^2}$$
   $$f_{xy} = (f_x)_y = -\frac{2x}{(x^2+2y^2)^2}\cdot 4y = \frac{-8xy}{(x^2+2y^2)^2}$$

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3. [**15 Points**] Find the equation of the tangent plane to the surface $x^2 + 2y^2z - 3z^2 = 3$ at the point $(2,\,-1,\,1)$.

▶ **Solution.** $f_x = 2x$, $f_y = 4yz$, and $f_z = (2y^2 - 6z)$, so at $(2,\,-1,\,1)$ these evaluate to $f_x = 4$, $f_y = -4$, $f_z = -4$. Thus, the equation of the tangent plane at this point is

$$4(x-2) - 4(y+1) - 4(z-1) = 0$$

or, equivalently

$$x - y - z = 2.$$

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4. [**15 Points**] Let $f(x,\,y) = x^2y$ where $x = e^{2t}$ and $y = 3s + 2t + 2$. Compute $\dfrac{\partial f}{\partial t}$ when $s = 0$ and $t = 0$.

▶ **Solution.** Use the chain rule.

$$\begin{aligned}\frac{\partial f}{\partial t} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}\\ &= 2xy \cdot 2e^{2t} + x^2 \cdot 2\\ &= 2e^{2t}(3s + 2t + 2)2e^{2t} + 2e^{4t}.\end{aligned}$$

Now evaluate at $s = 0$, $t = 0$, so that

$$\frac{\partial f}{\partial t}(0,\,0) = 2 \cdot 2 \cdot 2 + 2 = 10.$$

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5. [**20 Points**] Let $f(x,\,y) = x^2 + 2y^2 - 4x - 2y$.

(a) Find ∇f at $(2,\,1)$.

▶ **Solution.** $\nabla f = (2x - 4)\mathbf{i} + (4y - 2)\mathbf{j} = 0\mathbf{i} + 2\mathbf{j}$ at $(2,\,1)$. ◀

(b) Find the directional derivative of f at $(2,\,1)$ in the direction of the vector $\mathbf{v} = -\mathbf{i} + \mathbf{j}$.

▶ **Solution.** The unit direction vector in the direction of $\mathbf{v}$ is

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{-\mathbf{i} + \mathbf{j}}{\sqrt{(-1)^2 + 1^2}} = \frac{-\mathbf{i} + \mathbf{j}}{\sqrt{2}}.$$

Thus,

$$D_{\mathbf{u}}f(2,\,1) = 2\mathbf{j} \cdot \frac{-\mathbf{i} + \mathbf{j}}{\sqrt{2}} = \frac{2}{\sqrt{2}}.$$

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(c) In which direction is f increasing most rapidly at $(2,\, 1)$, and what is the maximum rate of increase of f at this point?

▶ **Solution.** The most rapid increase is in the direction of the gradient, that is, in the direction of $\mathbf{j}$. The maximum rate of increase of f at this point is $|\nabla f| = 2$. ◀

6. **[20 Points]** Let $f(x,\, y) = x^3 + y^3 + 3x^2 - 3y^2$.

(a) Find all the critical points of $f(x,\, y)$.

▶ **Solution.** Solve for the critical points:

$$\frac{\partial f}{\partial x} = 3x^2 + 6x = 0$$
$$\frac{\partial f}{\partial y} = 3y^2 + 6y = 0.$$

From the first equation, $x = 0$ or $x = -2$ and from the second equation $y = 0$ or $y = 2$. Thus, there are two critical points, $(0,\, 0)$, $(0,\, 2)$, $(-2,\, 0)$, and $(-2,\, 2)$. ◀

(b) Determine whether each critical point is a local minimum, local maximum, or saddle point.

▶ **Solution.** Use the second derivative test: Compute the second derivatives: $f_{xx} = 6x + 6$, $f_{xy} = 0$, and $f_{yy} = 6y - 6$. Thus, the discriminant is

$$D = f_{xx}f_{yy} - (f_{xy})^2 = (6x + 6)(6y - 6) = 36(x + 1)(y - 1).$$

Thus:

- At $(0,\, 0)$, the discriminant is $D(0, 0) = -36 < 0$. Hence, $(0,\, 0)$ is a saddle point.
- At $(0,\, 2)$, the discriminant is $D(0,\, 2) = 36 > 0$. Since $f_{xx}(0,\, 2) = 6 > 0$, it follows that $(0,\, 2)$ is a local minimum.
- At $(-2,\, 0)$, the discriminant is $D(-2,\, 0) = 36 > 0$. Since $f_{xx}(-2,\, 0) = -6 < 0$, it follows that $(-2,\, 0)$ is a local maximum.
- At $(-2,\, 2)$, the discriminant is $D(-2,\, 2) = -36 < 0$. Hence, $(-2,\, 2)$ is a saddle point.

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