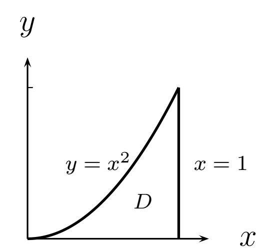
Instructions

- Answer each of the questions on your own paper, and be sure to show your work so that partial credit can be adequately assessed. Put your name on each page of your paper.
- You may use a scientific calculator, but it should not be needed. If you use a calculator you must still write out the operations done on the calculator to show that you know how to solve the problem.
- There are 5 problems, with a total of 100 points possible.

1. [**20 Points**] Compute $\iint_D x\cos y\,dA$ where D is the region bounded by $y = 0$, $y = x^2$, and $x = 1$.

▶ **Solution.** First draw the domain of integration D:



Thus,

$$D = \left\{(x,\,y) \mid 0 \le x \le 1,\; 0 \le y \le x^2\right\}$$

so

$$\begin{aligned}\iint_D x\cos y\,dA &= \int_0^1 \int_0^{x^2} x\cos y\,dy\,dx \\ &= \int_0^1 x\sin y\big|_{y=0}^{y=x^2}\,dx \\ &= \int_0^1 x\sin(x^2)\,dx \\ &= \frac{1}{2}\left(-\cos x^2\right)\big|_0^1 \\ &= \frac{1}{2}(1-\cos 1) \approx 0.2298.\end{aligned}$$

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2. [**20 Points**] Compute $\iiint_E y\,dV$ where E is the region defined by

$$E = \{(x,\,y,\,z) \mid 0 \le x \le 3,\; 0 \le y \le x,\; x - y \le z \le x + y\}\,.$$

▶ **Solution.**

$$\begin{aligned}\iiint_E y\,dV &= \int_0^3\int_0^x\int_{x-y}^{x+y} y\,dz\,dy\,dx = \int_0^3\int_0^x zy\big|_{z=x-y}^{z=x+y}\,dy\,dx\\ &= \int_0^3\int_0^x (xy+y^2)-(xy-y^2)\,dy\,dx = \int_0^3\int_0^x 2y^2\,dy\,dx = \int_0^3 \frac{2}{3}y^3\Big|_{y=0}^{y=x}\,dx\\ &= \int_0^3 \frac{2}{3}x^3\,dx = \frac{1}{6}x^4\Big|_0^3 = \frac{3^4}{6} = \frac{27}{2}.\end{aligned}$$

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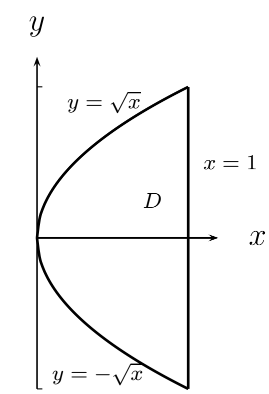
3. [**20 Points**] Evaluate the integral $\int_0^1\int_{-\sqrt{x}}^{\sqrt{x}} \frac{1}{1-y}\,dy\,dx$ by changing the order of integration.

▶ **Solution.** The domain of integration is $D = \{(x,\,y) \mid 0 \le x \le 1,\ -\sqrt{x} \le y \le \sqrt{x}\}$. Sketch this domain:



Thus, $D = \{(x,\,y) \mid -1 \le y \le 1,\ y^2 \le x \le 1\}$ and we can reverse the order of integration as

$$\begin{aligned}\int_0^1\int_{-\sqrt{x}}^{\sqrt{x}} \frac{1}{1-y}\,dy\,dx &= \iint_D \frac{1}{1-y}\,dA = \int_{-1}^1\int_{y^2}^1 \frac{1}{1-y}\,dx\,dy = \int_{-1}^1 \frac{x}{1-y}\Big|_{x=y^2}^{x=1}\,dy\\ &= \int_{-1}^1 \frac{1-y^2}{1-y}\,dy = \int_{-1}^1 (1+y)\,dy = \left(y+\frac{y^2}{2}\right)\Big|_{-1}^1 = 2.\end{aligned}$$

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4. [**20 Points**] Let D be the region in the first quadrant of the x, y-plane that lies between the two circles with center at the origin and radii 2 and 5:

$$D = \left\{(x,\,y) \mid 4 \le x^2+y^2 \le 25,\ x \ge 0, \text{ and } y \ge 0\right\}.$$

Write $\iint_D xy\,dx\,dy$ as an iterated integral in terms of polar coordinates. Then evaluate the integral, showing your procedure clearly.

▶ **Solution.** The region D is described in polar coordinates by

$$D = \left\{(r,\,\theta) \mid 0 \le \theta \le \frac{\pi}{2},\ 0 \le r \le 5\right\}.$$

Thus,

$$\begin{aligned}\iint_D xy\,dx\,dy &= \int_0^{\pi/2}\int_2^5 (r\cos\theta)(r\sin\theta)r\,dr\,d\theta = \int_0^{\pi/2}\int_2^5 r^3(\cos\theta\sin\theta)\,dr\,d\theta \\ &= \left(\int_0^{\pi/2}\cos\theta\sin\theta\,d\theta\right)\left(\int_2^5 r^3\,dr\right) = \left.\frac{\sin^2\theta}{2}\right|_0^{\pi/2}\cdot\left.\frac{r^4}{4}\right|_2^5 \\ &= \frac{1}{2}\frac{5^4-2^4}{4} = \frac{609}{8}.\end{aligned}$$

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5. **[20 Points]** Use cylindrical coordinates to find the volume of the region outside the cylinder $x^2+y^2=1$ and inside the paraboloid $z=4-x^2-y^2$, with $z \ge 0$.

▶ **Solution.** The region E is described in cylindrical coordinates by

$$E = \left\{(r,\,\theta,\,z) \mid 0 \le \theta \le 2\pi,\ 1 \le r \le 2,\ 0 \le z \le 4-r^2\right\}.$$

Thus,

$$\begin{aligned}V = \iiint_E dV &= \int_0^{2\pi}\int_1^2\int_0^{4-r^2} r\,dz\,dr\,d\theta \\ &= \int_0^{2\pi}\int_1^2 rz\Big|_{z=0}^{z=4-r^2}\,dr\,d\theta = \int_0^{2\pi}\int_1^2 r(4-r^2)\,dr\,d\theta \\ &= \int_0^{2\pi}\left.\left[2r^2-\frac{r^4}{4}\right]\right|_1^2 d\theta \\ &= \int_0^{2\pi}\left((8-4)-(2-\frac{1}{4})\right)d\theta = \int_0^{2\pi}\frac{9}{4}\,d\theta = \frac{9\pi}{2}.\end{aligned}$$

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