1. Evaluate the following limit or determine that it does not exist:

$$\lim_{(x,\,y)\to(0,\,2)} \left(\frac{1}{2xy} - \frac{1}{xy(x+2)}\right).$$

▶ **Solution.** Combine the two rational functions using a common denominator:

$$\lim_{(x,\,y)\to(0,\,2)} \left(\frac{1}{2xy} - \frac{1}{xy(x+2)}\right) = \lim_{(x,\,y)\to(0,\,2)} \frac{(x+2)-2}{2xy(x+2)} = \lim_{(x,\,y)\to(0,\,2)} \frac{1}{2y(x+2)} = \frac{1}{8}.$$

◀

2. Show that the function $w = \ln(x^2 + y^2)$ satisfies the two-dimensional Laplace equation $w_{xx} + w_{yy} = 0$.

▶ **Solution.** Compute the partial derivatives of w:

$$w_x = \frac{2x}{x^2+y^2} \qquad w_{xx} = \frac{2(x^2+y^2) - 2x(2x)}{(x^2+y^2)^2} = \frac{2y^2-2x^2}{(x^2+y^2)^2}$$

$$w_y = \frac{2y}{x^2+y^2} \qquad w_{yy} = \frac{2(x^2+y^2) - 2y(2y)}{(x^2+y^2)^2} = \frac{2x^2-2y^2}{(x^2+y^2)^2}.$$

Then

$$w_{xx} + w_{yy} = \frac{2y^2-2x^2}{(x^2+y^2)^2} + \frac{2x^2-2y^2}{(x^2+y^2)^2} = 0.$$

◀

3. Let $f(x,\,y) = 3xy^2 - x - y$.

(a) Find ∇f at $(2,\,1)$.

▶ **Solution.** $\nabla f = (3y^2 - 1)\mathbf{i} + (6xy - 1)\mathbf{j}$. Thus $\nabla f(2,\,1) = 2\mathbf{i} + 11\mathbf{j}$. ◀

(b) Find the equation of the tangent plane to the graph of $f(x,\,y)$ at the point $(2,\,1,\,3)$.

▶ **Solution.** $z - 3 = 2(x-2) - 11(y-1)$ or $z = 2x - 11y + 10$. ◀

(c) Use linear approximation to estimate the value of $f(2.1,\,0.9)$.

▶ **Solution.** $\Delta x = 2.1 - 2 = 1/10$ and $\Delta y = 0.9 - 1 = -1/10$. Thus

$$f(2.1,\,0.9) \approx f(2,\,1) + f_x(2,\,1)\Delta x + f_y(2,\,1)\Delta y = 3 + 2(1/10) + 11(-1/10) = 2.1.$$

◀

(d) Find the directional derivative of f at $(2,\,1)$ in the direction $\mathbf{i} + \mathbf{j}$.

▶ **Solution.** The direction vector $\mathbf{u}$ in the direction $\mathbf{i} + \mathbf{j}$ is $\mathbf{u} = (\mathbf{i} + \mathbf{j})/\sqrt{2}$ so

$$D_{\mathbf{u}} f(2,\, 1) = \nabla f(2,\, 1) \cdot \mathbf{u} = (2 + 11)/\sqrt{2} = \frac{13}{\sqrt{2}}.$$

◀

4. Let $w = f(u,\, v)$ where $u = x^2 - y^2$ and $v = 2xy$.

   (a) Using the chain rule, express

   $$\frac{\partial w}{\partial x} \qquad \text{and} \qquad \frac{\partial w}{\partial y}$$

   in terms of x, y, f_u and f_v.

   ▶ **Solution.**

   $$\begin{aligned} \frac{\partial w}{\partial x} &= f_u \frac{\partial u}{\partial x} + f_v \frac{\partial v}{\partial x} \\ &= 2x f_u + 2y f_v, \end{aligned}$$

   and

   $$\begin{aligned} \frac{\partial w}{\partial y} &= f_u \frac{\partial u}{\partial y} + f_v \frac{\partial v}{\partial y} \\ &= -2y f_u + 2x f_v, \end{aligned}$$

   ◀

   (b) Assume that w solves the differential equation $uw_u + vw_v = 1$. Find $xw_x + yw_y$.

   ▶ **Solution.** We have

   $$1 = uw_u + vw_v = (x^2 - y^2)w_v + 2xyw_v.$$

   Thus

   $$\begin{aligned} xw_x + yw_y &= x(2xw_u + 2yw_v) + y(-2yw_u + 2xw_v) \\ &= 2x^2 w_u + 4xyw_v - 2y^2 w_u \\ &= 2(x^2 - y^2)w_u + 4xyw_v \\ &= 2uw_u + 2vw_v \\ &= 2. \end{aligned}$$

   ◀

5. Suppose that the equation $x^2yz^3 + xz^2 + z + 2 = 0$ has a differentiable solution for z as a function of x and y. Use implicit differentiation to find $\dfrac{\partial z}{\partial x}$ at the point $(1,\, 2,\, -1)$.

► **Solution.** Let $f(x, y, z) = x^2yz^3 + xz^2 + z + 2$. Then

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_y} = -\frac{2xyz^3 + z^2}{3x^2yz^2 + 2xz + 1} = \frac{3}{5}$$

at the indicated point $(1, 2, -1)$. ◄

6. Let $f(x, y) = -3x^2 - 4xy - y^2 - 12y + 16x$.

   (a) Find all the critical points of $f(x, y)$, and determine what type each critical point is.

   ► **Solution.**

$$\begin{aligned} f_x &= -6x - 4y + 16 = 0 \qquad &\Longrightarrow& \qquad -3x - 2y + 8 = 0 \\ f_y &= -4x - 2y - 12 = 0 \qquad &\Longrightarrow& \qquad 4x + 2y + 12 = 0 \end{aligned}$$

   Add the last two equations to get $x = -20$ and substitute in either of the equations to get $y = 34$. Therefore, there is just one critical point at $(-20, 34)$. Since the discriminant is

$$D = w_{xx}w_{yy} - w_{xy}^2 = (-6)(-2) - (-4)^2 = 12 - 16 = -4 < 0,$$

   the critical point is a saddle point. ◄

   (b) Find the global maximum and minimum of $f(x, y)$ over the triangle with sides $x = 0$, $y = 0$ and $y + x = 4$.

   ► **Solution.** Since the only critical point of w is not included in the inside of the given triangle, the global maximum and minimum must be on the boundary of the triangle. The boundary consists of 3 line segments:

   - $x = 0$ and $0 \le y \le 4$ on which $w = -y^2 - 12y$. It has a maximum of $w = 0$ at $y = 0$ and a minimum of $w = -64$ at $y = 4$.
   - $y = 0$ and $0 \le x \le 4$ on which $w = -3x^2 + 16x$. The graph is a parabola pointing down, so it has a maximum at the vertex given by $w_x = -6x + 16 = 0 \implies x = 8/3$. Thus, the maximum value of w on this line segment is $w = -3(8/3)^2 + 16 \cdot 8/3 = 64/3$.
   - The third leg of the triangle has the equation $y = 4 - x$ for $0 \le x \le 4$. Substituting this value for y into the equation for w gives that

$$\begin{aligned} w &= -3x^2 - 4x(4 - x) - (4 - x)^2 - 12(4 - x) + 16x \\ &= -3x^2 - 16x + 4x^2 - (x^2 - 8x + 16) - 48 + 12x + 16x \\ &= 20x - 64. \end{aligned}$$

     This is linear with positive slope so the minimum occurs when $x = 0$, which gives $w = -64$, and the maximum occurs when $x = 4$ which gives $w = 16$.

Thus, we see that the minimum on the triangle occurs at $(0, 4)$ where $w = -64$ and the maximum occurs at $(8/3, 0)$ where $w = 64/3$. ◀

7. Use Lagrange Multipliers to find the global *maximum and minimum* values of $f(x, y) = x^2 + 2y^2 - 4y$ subject to the constraint $x^2 + y^2 = 9$.

► **Solution.** Let $g(x, y) = x^2 + y^2$. Then solve the equations $\nabla f(x, y) = \lambda \nabla g(x, y)$, $g(x, y) = k$ to find the critical points:

$$\begin{aligned} 2x &= \lambda 2x \\ 4y - 4 &= \lambda 2y \\ x^2 + y^2 &= 9. \end{aligned}$$

The first equation gives $x(\lambda - 1) = 0$ so $x = 0$ or $\lambda = 1$. If $x = 0$, then the third equation gives $y^2 = 9$ so $y = \pm 3$. If $\lambda = 1$ then the second equation gives $4y - 4 = 2y$ so $y = 2$. The third equation then gives $x^2 + 2^2 = 9$ so $x = \pm\sqrt{5}$. Thus, there are four critical points: $(0, \pm 3)$ and $(\pm\sqrt{5}, 2)$. Evaluating $f(x, y)$ at these points gives: $f(0, 3) = 6$, $f(0, -3) = 30$, $f(\sqrt{5}, 2) = 5 = f(-\sqrt{5}, 2)$. Therefore, the global maximum is 30, which occurs at $(0, -3)$, and the global minimum is 5, which occurs at the two points $(\pm\sqrt{5}, 2)$. ◀