1. Compute 
$$\int_0^2 \int_{y^2}^{2y} (4x - y) \, dx \, dy.$$

► Solution.

$$\int_{0}^{2} \int_{y^{2}}^{2y} (4x - y) \, dx \, dy = \int_{0}^{2} (2x^{2} - xy) \Big|_{x=y^{2}}^{x=2y} \, dy$$
$$= \int_{0}^{2} (6y^{2} - 2y^{4} + y^{3}) \, dy = \left(2y^{3} - \frac{2y^{5}}{5} + \frac{y^{4}}{4}\right) \Big|_{0}^{2}$$
$$= 16 - \frac{64}{5} + 4 = \frac{36}{5}.$$

- 2. Compute  $\int_0^1 \int_0^x \int_0^{xy} xyz \, dz \, dy \, dx.$ 
  - ► Solution.

$$\int_{0}^{1} \int_{0}^{x} \int_{0}^{xy} xyz \, dz \, dy \, dx = \int_{0}^{1} \int_{0}^{x} \frac{xyz^{2}}{2} \Big|_{z=0}^{z=xy} \, dy \, dx$$
$$= \int_{0}^{1} \int_{0}^{x} \frac{x^{3}y^{3}}{2} \, dy \, dx = \int_{0}^{1} \frac{x^{3}y^{4}}{8} \Big|_{y=0}^{y=x} \, dx$$
$$= \int_{0}^{1} \frac{x^{7}}{7} \, dx = \frac{x^{8}}{64} \Big|_{0}^{1} = \frac{1}{64}.$$

3. Let R be the region in the plane bounded by the graphs of y<sup>2</sup> = 4 + x and y<sup>2</sup> = 4 - x.
(a) Sketch R.



- (b) If f(x, y) is an arbitrary continuous function defined on R, express  $\iint_R f(x, y) dA$  as an iterated double integral.
  - ► Solution.

$$\iint_R f(x, y) \, dA = \int_{-2}^2 \int_{y^2 - 4}^{4 - y^2} f(x, y) \, dx \, dy$$

4. Compute the following integral:

$$\int_0^1 \int_{\sqrt{y}}^1 \sin(x^3) \, dx \, dy.$$

(*Hint:* First draw the domain of integration. Then reverse the order of integration.)

**Solution.** First draw the domain of integration R:



Then write the curve  $x = \sqrt{y}$  as  $y = x^2$  and change the order of integration

$$\int_0^1 \int_{\sqrt{y}}^1 \sin(x^3) \, dx \, dy = \int_0^1 \int_0^{x^2} \sin(x^3) \, dy \, dx = \int_0^1 y \sin(x^3) \Big|_0^{x^2} \, dx$$
$$= \int_0^1 x^2 \sin(x^3) \, dx = -\frac{1}{3} \cos(x^3) \Big|_0^1 = \frac{1}{3} (1 - \cos 1).$$

5. Compute the following integral:

$$\int_0^1 \int_y^{\sqrt{2-y^2}} \sqrt{x^2 + y^2} \, dx \, dy.$$

(*Hint:* First draw the domain of integration. Then use polar coordinates.)

 $\blacktriangleright$  Solution. First draw the domain of integration R:



Then R can be expressed in polar coordinates as  $0 \le \theta \le \pi/4$ ,  $0 \le r \le \sqrt{2}$ . Then  $dA = rdr d\theta$  and

$$\int_{0}^{1} \int_{y}^{\sqrt{2-y^{2}}} \sqrt{x^{2}+y^{2}} \, dx \, dy = \iint_{R} \sqrt{x^{2}+y^{2}} \, dA$$
$$= \int_{0}^{\pi/4} \int_{0}^{\sqrt{2}} r \, r \, dr \, d\theta = \int_{0}^{\pi/4} \int_{0}^{\sqrt{2}} r^{2} \, dr \, d\theta$$
$$= \int_{0}^{\pi/4} \frac{r^{3}}{3} \Big|_{r=0}^{r=\sqrt{2}} d\theta = \int_{0}^{\pi/4} \frac{2\sqrt{2}}{3} \, d\theta$$
$$= \frac{2\sqrt{2}\pi}{12} = \frac{\sqrt{2}\pi}{6}.$$

- 6. Compute the area of one leaf of the four leaved rose  $r = a \sin(2\theta)$ .
  - ► Solution. First draw a picture of one leaf:



Then the single leaf can be expressed in polar coordinates as  $0 \le \theta \le \frac{\pi}{2}$ ,  $0 \le r \le a \sin 2\theta$ , and the area is given by

$$Area = \iint_{R} dA = \int_{0}^{\pi/2} \int_{0}^{a \sin 2\theta} r \, dr \, d\theta = \int_{0}^{\pi/4} \frac{r^{2}}{2} \Big|_{0}^{a \sin 2\theta} d\theta$$
$$= \frac{a^{2}}{2} \int_{0}^{\pi/2} \sin^{2} 2\theta \, d\theta = \frac{a^{2}}{2} \int_{0}^{\pi/2} \frac{1 - \cos 4\theta}{2} \, d\theta$$
$$= \frac{a^{2}}{4} \left( \theta - \frac{\sin 4\theta}{4} \right) \Big|_{0}^{\pi/2}$$
$$= \frac{a^{2}\pi}{8}.$$

- •
- 7. Compute the volume of the region in the first octant that is bounded by the coordinate planes and the plane x + y + z = 3.

▶ Solution. The region is above the triangular region in the *xy*-plane bounded by the axes and the line x + y = 3, and it is below the plane z = 3 - x - y. Thus, the volume is given by

$$\int_{0}^{3} \int_{0}^{3-y} (3-x-y) \, dx \, dy = \int_{0}^{3} \left( 3x - \frac{x^{2}}{2} - xy \right) \Big|_{x=0}^{x=3-y} \, dy$$
$$= \int_{0}^{3} \left( 9 - 3y - \frac{9 - 6y + y^{2}}{2} - (3y - y^{2}) \right) \, dy$$
$$= \int_{0}^{3} \left( \frac{9}{2} + \frac{y^{2}}{2} - 3y \right) \, dy = \left( \frac{9}{2}y + \frac{y^{3}}{6} - \frac{3y^{2}}{2} \right) \Big|_{0}^{3}$$
$$= \frac{27}{2} + \frac{27}{6} - \frac{27}{2} = \frac{27}{6} = \frac{9}{2}.$$

- 8. Compute the volume of the finite region Q bounded by the graphs of  $z = 9 x^2 y^2$ ,  $x^2 + y^2 = 4$ , and z = 0. Use cylindrical coordinates.
  - **Solution.** The volume of Q is

$$\iiint_Q dV = \int_0^{2\pi} \int_0^2 \int_0^{9-r^2} r \, dz \, dr \, d\theta$$
  
=  $\int_0^{2\pi} \int_0^2 z |_{z=0}^{z=9-r^2} r \, dr \, d\theta = \int_0^{2\pi} \int_0^2 (9r - r^3) \, dr \, d\theta$   
=  $\int_0^{2\pi} \left( \frac{9r^2}{2} - \frac{r^4}{4} \right) \Big|_0^2 \, d\theta = \int_0^{2\pi} 14 \, d\theta$   
=  $28\pi$ .

-	
-	
-	

9. Let Q be the region bounded below by the cone  $z^2 = x^2 + y^2$  and above by the sphere of radius  $\sqrt{2}$  and center at the origin. Compute the volume of Q using spherical coordinates.

▶ Solution. The cone and the sphere intersect when  $x^2 + y^2 = z^2 = 2 - x^2 - y^2$ so  $x^2 + y^2 = 1$ . In Q the z-coordinate is positive. The cone is defined in spherical coordinates by  $\phi = \pi/4$  and Q is symmetric around the z-axis. Thus, Q is defined in spherical coordinates by  $0 \le \rho \le \sqrt{2}$ ,  $0 \le \theta \le 2\pi$ ,  $0 \le \phi \le \pi/4$ . Hence, the volume of Q is given by the integral

$$\iiint_{Q} dV = \int_{0}^{\frac{\pi}{4}} \int_{0}^{2\pi} \int_{0}^{\sqrt{2}} \rho^{2} \sin \phi \, d\rho \, d\theta \, d\phi$$
$$= \int_{0}^{\frac{\pi}{4}} \int_{0}^{2\pi} \frac{\rho^{3}}{3} \sin \phi \Big|_{0}^{\sqrt{2}} d\theta \, d\phi = \frac{2\sqrt{2}}{3} \int_{0}^{\frac{\pi}{4}} \int_{0}^{2\pi} \sin \phi \, d\theta \, d\phi$$
$$= (2\pi) \left(\frac{2\sqrt{2}}{3}\right) \int_{0}^{\frac{\pi}{4}} \sin \phi \, d\phi = (2\pi) \left(\frac{2\sqrt{2}}{3}\right) (-\cos \phi) \Big|_{0}^{\frac{\pi}{4}}$$
$$= \frac{4\pi}{3} (\sqrt{2} - 1).$$

Math	2057
maun	2001

◀