1. Use Green's theorem to evaluate the line integral $\int_C (x^3 - y^3)\,dx + (x^3 + y^3)\,dy$ if C is the circle $x^2 + y^2 = 4$ oriented counter-clockwise.

   ▶ **Solution.** C is the boundary of the disc $D = \{(x, y) \mid x^2 + y^2 \le 4\}$ so Green's theorem gives

$$\begin{aligned}\int_C (x^3 - y^3)\,dx + (x^3 + y^3)\,dy &= \iint_D (3x^2 + 3y^2)\,dA \qquad \text{convert to polar coordinates}\\ &= \int_0^{2\pi}\int_0^2 3r^2\,r\,dr\,d\theta = \int_0^{2\pi} \frac{3}{4}r^4\Big|_0^2\,d\theta\\ &= \int_0^{2\pi} 12\,d\theta = 24\pi.\end{aligned}$$

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2. Let $\mathbf{F}(x, y) = (3x^2 - 6y^2)\mathbf{i} + (-12xy + 4y)\mathbf{j}$.

   (a) Show that $\mathbf{F}$ is conservative.

   ▶ **Solution.**
$$\frac{\partial}{\partial x}(-12xy + 4y) = -12y = \frac{\partial}{\partial y}(3x^2 - 6y^2)$$
   Hence, F is conservative. ◀

   (b) Find a function $f(x, y)$ such that $\mathbf{F} = \nabla f$.

   ▶ **Solution.** We need $f(x, y)$ with $f_x = 3x^2 - 6y^2$ and $f_y = -12xy + 4y$. Integrate the first equation with respect to x to get $f(x, y) = x^3 - 6xy^2 + g(y)$ where $g(y)$ is some function of y. Differentiate with respect to y to get

$$-12xy + g'(y) = f_y = -12xy + 4y.$$

   Thus $g'(y) = 4y$ so $g(y) = 2y^2 + c$ where c is an arbitrary constant. Letting $c = 0$ gives a potential function $f(x, y) = x^3 - 6xy^2 + 2y^2$. ◀

   (c) Let C be the curve consisting of the straight line from $(1, 2)$ to $(-1, 2)$ followed by the straight line from $(-1, 2)$ to $(2, -1)$. Calculate $\int_C \mathbf{F}\cdot d\mathbf{r}$.

   ▶ **Solution.** Since $\mathbf{F} = \nabla f$ apply the fundamental theorem for line integrals to get

$$\int_C \mathbf{F}\cdot d\mathbf{r} = f(2, -1) - f(1, 2) = (8 - 12 + 2) - (1 - 24 + 8) = 13.$$

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