Instructions. Answer each of the questions on your own paper. Put your name on each page of your paper. Be sure to show your work so that partial credit can be adequately assessed. *Credit will not be given for answers (even correct ones) without supporting work.* As usual, a copy of the Table of Laplace transforms from the text will be provided.

In Exercises 1 – 6, solve the given differential equation. If initial values are given, solve the initial value problem. Otherwise, give the general solution. Some problems may be solvable by more than one technique. You are free to choose whatever technique that you deem to be most appropriate.

1. [**12 Points**] $y' - \dfrac{y}{t} = 2t^2, \quad y(1) = -2.$
2. [**12 Points**] $y' = 3t^2y^2 - 4ty^2, \quad y(0) = -2.$
3. [**10 Points**] $y'' - 3y' + 2y = 0, \quad y(0) = 1,\ y'(0) = 3.$
4. [**12 Points**] $y'' - 3y' + 2y = 3e^{2t}$
5. [**9 Points**] $y'' + 6y' + 25y = 0$
6. [**9 Points**] $4y'' + 12y' + 9y = 0$
7. [**12 Points**] Find a particular solution of the differential equation

$$y'' + \frac{1}{t}y' - \frac{1}{t^2}y = 72t^3$$

given the fact that the general solution of the associated homogeneous equation is

$$y_h = c_1 t + c_2 t^{-1}.$$

8. [**14 Points**] Consider the initial value problem

$$y'' + y = f(t), \qquad y(0) = 3,\ y'(0) = -1$$

where

$$f(t) = \begin{cases} 3 & \text{for } 0 \le t < 5, \\ 2 & \text{for } t \ge 5. \end{cases}$$

   (a) Find $F(s)$, the Laplace transform of $f(t)$.

   (b) Find $Y(s)$, the Laplace transform of $y(t)$. (Note: You may express your answer in terms of $F(s)$.) **Do not** solve for $y(t)$.

9. [**14 Points**] Find the Laplace transform of each of the following functions.

   (a) $f(t) = (t-3)^2(h(t-2) - h(t-4))$

   (b) $g(t) = (t^2 - \cos 2t)e^{-3t}$

10. [**14 Points**] Compute each of the following inverse Laplace transforms.

    (a) $\mathcal{L}^{-1}\left\{\dfrac{s-10}{s^2+s-2}\right\}$

    (b) $\mathcal{L}^{-1}\left\{\dfrac{e^{-2s}}{s^2+9}\right\}$

11. [**20 Points**] Let $A = \begin{bmatrix} -1 & -1 \\ 9 & -1 \end{bmatrix}$.

    (a) Compute $(sI - A)^{-1}$.

    (b) Find $e^{At} = \mathcal{L}^{-1}\{(sI - A)^{-1}\}$.

    (c) Find the general solution of the system $\mathbf{y}' = A\mathbf{y}$.

    (d) Solve the initial value problem $\mathbf{y}' = A\mathbf{y}$, $\mathbf{y}(0) = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$.

12. [**12 Points**] A tank, which has a capacity of 100 gallons, initially contains 10 gallons of pure water. Brine (a water-salt mixture) containing 3 pounds of salt per gallon flows into the tank at the rate of 4 gal/min, and the mixture (which is assumed to be perfectly mixed) flows out of the tank at the rate of 2 gal/min.

    (a) How long will it take for the tank to begin to overflow?

    (b) If $y(t)$ denotes the number of pounds of salt in the tank at time t, and T denotes the time found in part (a) when the tank begins to overflow, then write down the differential equation satisfied by $y(t)$ for $t \in [0, T]$. Do **not** solve this equation.

    (c) What is the initial value $y(0)$?