Instructions. Answer each of the questions on your own paper. Put your name on each page of your paper. Be sure to show your work so that partial credit can be adequately assessed. *Credit will not be given for answers (even correct ones) without supporting work.* As usual, a copy of the Table of Laplace transforms from the text will be provided.

In Exercises 1 – 6, solve the given differential equation. If initial values are given, solve the initial value problem. Otherwise, give the general solution. Some problems may be solvable by more than one technique. You are free to choose whatever technique that you deem to be most appropriate.

1. [**12 Points**] $y' - \dfrac{y}{t} = 2t^2, \quad y(1) = -2.$

   ▶ **Solution.** This equation is linear with coefficient function $p(t) = -1/t$ so that $P(t) = \int p(t)\,dt = \ln t = \ln(t^{-1})$; thus an integrating factor $\mu(t)$ is given by

   $$\mu(t) = e^{P(t)} = e^{\ln(t^{t^{-1}})} = t^{-1}.$$

   Multiplying the equation by $\mu(t)$ makes the left hand side a perfect derivative:

   $$\frac{d}{dt}(t^{-1}y) = t^{-1}2t^2 = 2t,$$

   and integrating gives $t^{-1}y = t^2 + C$. Solving for y gives $y = t^3 + Ct$, and substituting the initial condition $y(1) = -2$ gives

   $$-2 = y(1) = 1 + C \implies C = -3,$$

   so that the solution of the initial value problem is

   $$\boxed{y = t^3 - 3t.}$$

   ◀

2. [**12 Points**] $y' = 3t^2y^2 - 4ty^2, \quad y(0) = -2.$

   ▶ **Solution.** This equation is separable since $3t^2y^2 - 4ty^2 = (3t^2 - 4t)y^2$. Thus, separating the variables and writing the equation in differential form gives

   $$y^{-1}\,dy = (3t^2 - 4t)\,dt,$$

   and integration then gives

   $$\frac{-1}{y} = t^3 - 2t^2 + C.$$

   Taking the reciprocal of both sides:

   $$y = \frac{-1}{t^2 - 2t^2 + C}.$$

Now solve for C from the initial value $y(0) = -2$:

$$-2 = y(0) = \frac{-1}{C} \implies C = \frac{1}{2}.$$

Hence

$$\boxed{y = \frac{-1}{t^3 - 2t^2 + \frac{1}{2}} = \frac{-2}{2t^3 - 4t^2 + 1}.}$$

◀

3. [**10 Points**] $y'' - 3y' + 2y = 0, \quad y(0) = 1,\ y'(0) = 3.$

▶ **Solution.** The characteristic polynomial is $p(s) = s^2 - 3s + 2 = (s-2)(s-1)$ which has the two simple roots $s = 1$ and $s = 2$. Thus, the general solution of the homogeneous equation is $y = c_1e^{2t} + c_2e^t$, and the constants c_1, c_2 are computed from the initial conditions

$$\begin{aligned} 1 &= y(0) = c_1 + c_2 \\ 3 &= y'(0) = 2c_1 + c_2, \end{aligned}$$

which are solved to give $c_1 = 2$, $c_1 = -1$. Thus

$$\boxed{y = 2e^{2t} - e^t.}$$

◀

4. [**12 Points**] $y'' - 3y' + 2y = 3e^{2t}$

▶ **Solution.** Use the method of undetermined coefficients. As in the previous exercise, the characteristic polynomial is $p(s) = s^2 - 3s + 2 = (s-2)(s-1)$. The Laplace transform of the right hand side is $3/(s-2)$ which has denominator $Q(s) = s-2$. Hence, $p(s)Q(s) = (s-2)^2(s-1)$, which determines the fundamental basis $\{e^t,\ e^{2t},\ te^{2t}\}$. Since the first two terms come from the homogeneous equation $y'' - 3y' + 2y = 0$, we conclude that the non-homogeneous equation has the form $y_p = Ate^{2t}$. Compute the derivatives of y_p:

$$\begin{aligned} y_p &= Ate^{2t} \\ y_p' &= 2Ate^{2t} + Ae^{2t} \\ y_p'' &= 4Ate^{2t} + 4Ae^{2t}, \end{aligned}$$

and substitute them into the equation

$$\begin{aligned} 3e^{2t} &= y_p'' - 3y_p' + 2y_p \\ &= 4Ate^{2t} + 4Ae^{2t} - 3(2Ate^{2t} + Ae^{2t}) + 2Ate^{2t} \\ &= Ae^{2t}. \end{aligned}$$

Hence $A = 3$ and a particular solution is $y_p = 3te^{2t}$, while the general solution is

$$\boxed{y = y_h + y_p = c_1e^{2t} + c_2e^t + 3te^{2t}.}$$

◀

5. [**9 Points**] $y'' + 6y' + 25y = 0$

   ▶ **Solution.** The characteristic polynomial is $p(s) = s^2 + 6s + 25 = (s+3)^2 + 16$ which has the pair of complex conjugate roots $s = -3 \pm 4i$. This gives

   $$\boxed{y = c_1 e^{-3t} \cos 4t + c_2 e^{-3t} \sin 4t.}$$

   ◀

6. [**9 Points**] $4y'' + 12y' + 9y = 0$

   ▶ **Solution.** The characteristic polynomial is $p(s) = 4s^2 + 12s + 9 = (2s+3)^2$ which has a single root $s = -3/2$ of multiplicity 2. This gives

   $$\boxed{y = c_1 e^{-3t/2} + c_2 t e^{-3t/2}.}$$

   ◀

7. [**12 Points**] Find a particular solution of the differential equation

   $$y'' + \frac{1}{t}y' - \frac{1}{t^2}y = 72t^3$$

   given the fact that the general solution of the associated homogeneous equation is

   $$y_h = c_1 t + c_2 t^{-1}.$$

   ▶ **Solution.** Use variation of parameters. From the knowledge of y_h we search for a particular solution of the form

   $$y_p = u_1 t + u_2 t^{-1},$$

   where u_1 and u_2 are unknown functions whose derivatives satisfy the equations

   $$\begin{aligned} u_1' t + u_2' t^{-1} &= 0 \\ u_1 - u_2 t^{-2} &= 72t^3. \end{aligned}$$

   Multiplying the first equation by t^{-1} and adding to the second equation gives $2u_1' = 72t^3$ so that $u_1' = 36t^3$. The first equation then gives

   $$u_2' = -t^2 u_1' = -36t^5,$$

   and integration gives

   $$u_1 = 9t^4 \quad \text{and} \quad u_2 = -6t^6,$$

   so that

   $$\boxed{y_p = u_1 t + u_2 t^{-1} = 9t^5 - 6t^5 = 3t^5.}$$

   ◀

8. [**14 Points**] Consider the initial value problem

$$y'' + y = f(t), \qquad y(0) = 3,\ y'(0) = -1$$

where

$$f(t) = \begin{cases} 3 & \text{for } 0 \le t < 5, \\ 2 & \text{for } t \ge 5. \end{cases}$$

(a) Find $F(s)$, the Laplace transform of $f(t)$.

▶ **Solution.** Write $f(t)$ in terms of the Heaviside step functions as follows:

$$\begin{aligned} f(t) &= 3\chi_{[0,\,5)}(t) + 2\chi_{[5,\,\infty)} \\ &= 3(h(t-0) - h(t-5)) + 2h(t-5) \\ &= 3 - h(t-5). \end{aligned}$$

Thus,

$$\boxed{F(s) = \frac{3}{s} - \frac{e^{-5s}}{s}.}$$

◀

(b) Find $Y(s)$, the Laplace transform of $y(t)$. (Note: You may express your answer in terms of $F(s)$.) **Do not** solve for $y(t)$.

▶ **Solution.** Applying the Laplace transform to both sides of the equation gives $s^2Y(s) - 3s + 1 + Y(s) = F(s)$, and solving for $Y(s)$ gives

$$\boxed{Y(s) = \frac{3s - 1 + F(s)}{s^2 + 1}.}$$

◀

9. [**14 Points**] Find the Laplace transform of each of the following functions.

(a) $f(t) = (t-3)^2(h(t-2) - h(t-4))$

▶ **Solution.** First write $f(s) = (t-3)^2h(t-2) - (t-3)^2h(t-4)$ and then apply the second translation formula to get

$$\begin{aligned} F(s) &= e^{-2s}\mathcal{L}\left\{(t+2-3)^2\right\} - e^{-4s}\mathcal{L}\left\{(t+4-3)^2\right\} \\ &= e^{-2s}\mathcal{L}\left\{t^2 - 2t + 1\right\} - e^{-4s}\mathcal{L}\left\{t^2 + 2t + 1\right\} \\ &= \boxed{e^{-2s}\left(\frac{2}{s^3} - \frac{2}{s^2} + \frac{1}{s}\right) - e^{-4s}\left(\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}\right).} \end{aligned}$$

◀

(b) $g(t) = (t^2 - \cos 2t)e^{-3t}$

▶ **Solution.** Using the first translation formula,

$$\boxed{G(s) = \frac{2}{(s+3)^3} - \frac{s+3}{(s+3)^2+4}.}$$

◀

10. [**14 Points**] Compute each of the following inverse Laplace transforms.

(a) $\mathcal{L}^{-1}\left\{\dfrac{s-10}{s^2+s-2}\right\}$

▶ **Solution.** Use partial fractions to write

$$\frac{s-10}{s^2+s-2} = \frac{4}{s+2} - \frac{3}{s-1}.$$

This gives

$$\boxed{\mathcal{L}^{-1}\left\{\frac{s-10}{s^2+s-2}\right\} = 4e^{-2t} - 3e^{t}.}$$

◀

(b) $\mathcal{L}^{-1}\left\{\dfrac{e^{-2s}}{s^2+9}\right\}$

▶ **Solution.** Since $\mathcal{L}^{-1}\left\{1\right\} s^2+9 = \dfrac{1}{3}\sin 3t$, the second translation formula gives

$$\boxed{\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2+9}\right\} = \frac{1}{3}(\sin 3(t-2))h(t-2).}$$

◀

11. [**20 Points**] Let $A = \begin{bmatrix} -1 & -1 \\ 9 & -1 \end{bmatrix}$.

(a) Compute $(sI-A)^{-1}$.

▶ **Solution.** First note that

$$p(s) = \{(\}\, sI - A) = \det \begin{bmatrix} s+1 & 1 \\ -9 & s+1 \end{bmatrix} = (s+1)^2 + 9.$$

Then the adjoint formula for the inverse of a matrix gives

$$(sI-A)^{-1} = \begin{bmatrix} \dfrac{s+1}{(s+1)^2+9} & \dfrac{-1}{(s+1)^2+9} \\ \dfrac{9}{(s+1)^2+9} & \dfrac{s+1}{(s+1)^2+9} \end{bmatrix}.$$

◀

(b) Find $e^{At} = \mathcal{L}^{-1}\left\{(sI - A)^{-1}\right\}$.

▶ **Solution.**

$$e^{At} = \mathcal{L}^{-1}\left\{(sI - A)^{-1}\right\} = \begin{bmatrix} e^{-t}\cos 3t & -\frac{1}{3}e^{-t}\sin 3t \\ 3e^{-t}\sin 3t & e^{-t}\cos 3t \end{bmatrix}.$$

◀

(c) Find the general solution of the system $\mathbf{y}' = A\mathbf{y}$.

▶ **Solution.** The general solution is

$$\mathbf{y} = e^{At}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} c_1 e^{-t}\cos 3t - \frac{1}{3}c_2 e^{-t}\sin 3t \\ 3c_1 e^{-t}\sin 3t + c_2 e^{-t}\cos 3t \end{bmatrix},$$

where $\mathbf{y}(0) = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$. ◀

(d) Solve the initial value problem $\mathbf{y}' = A\mathbf{y}$, $\mathbf{y}(0) = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$.

▶ **Solution.** Let $c_1 = 0$ and $c_2 = 3$ in the previous part. Hence

$$\mathbf{y}(t) = \begin{bmatrix} -e^{-t}\sin 3t \\ 3e^{-t}\cos 3t \end{bmatrix}.$$

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12. [**12 Points**] A tank, which has a capacity of 100 gallons, initially contains 10 gallons of pure water. Brine (a water-salt mixture) containing 3 pounds of salt per gallon flows into the tank at the rate of 4 gal/min, and the mixture (which is assumed to be perfectly mixed) flows out of the tank at the rate of 2 gal/min.

(a) How long will it take for the tank to begin to overflow?

▶ **Solution.** The volume of mixture in the tank is increasing at the rate of $4-2 = 2$ gal/min. So the volume at time t is given by $V(t) = 2t + 10$ because, there were initially 10 gallons of water in the tank. The tank is full when $V(t) = 100$, i.e., when $2t + 10 = 100$. Thus the tank is full after $T = 45$ min. ◀

(b) If $y(t)$ denotes the number of pounds of salt in the tank at time t, and T denotes the time found in part (a) when the tank begins to overflow, then write down the differential equation satisfied by $y(t)$ for $t \in [0, T]$. Do **not** solve this equation.

▶ **Solution.** Use the balance equation $y' = \text{Rate in} - \text{Rate out}$ for the rate of change of $y(t)$ as a function of time t. Now

$$\text{Rate in} = 4\text{gal/min} \times 3\text{lb/gal} = 12\text{lb/min},$$

while

$$\begin{aligned} \text{Rate out} &= 2\text{gal/min} \times \frac{y(t)}{V(t)}\text{lb/gal} \\ &= \frac{2y(t)}{10+2t}\text{lb/min} \\ &= \frac{y(t)}{5+t}\text{lb/min.} \end{aligned}$$

Hence, the differential equation satisfied by $y(t)$ is

$$\boxed{y' = 12 - \frac{y}{5+t}.}$$

◀

(c) What is the initial value $y(0)$?

▶ **Solution.** The mixture in the tank is initially pure water so

$$\boxed{y(0) = 0.}$$

◀