Instructions. Answer each of the questions on your own paper. Be sure to show your work so that partial credit can be adequately assessed. *Credit will not be given for answers (even correct ones) without supporting work.* Put your name on each page of your paper. A short table of Laplace Transforms and a table of common exact values of trigonometric functions is included on Page 2.

1. [16 Points Each] Solve each of the following initial value problems. Be sure to show all of your work.

   (a) $y' + \dfrac{t}{y} = 0, \quad y(1) = -3$

   ▶ **Solution.** This equation is separable. After separating variables, it becomes $yy' = -t$, which in differential form is $y\,dy = -t\,dt$ and integration then gives $y^2/2 = -t^2/2 + C$, so that any solution of the differential equation is an implicit solution of the algebraic equation $y^2 + t^2 = 2C$. The initial condition $y(1) = -3$ means that $y = -3$ when $t = 1$. This gives $(-3)^2 + 1^2 = 2C$ so $C = 5$. Hence the solution of the differential equation is an implicit solution of the equation $y^2 + t^2 = 10$, so that $y = \pm\sqrt{10 - t^2}$. Since $y(1) = -3$ is negative, it is necessary to take the negative sign, so that the solution of the initial value problem is

   $$\boxed{y = -\sqrt{10 - t^2}.}$$

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   (b) $y' - 2y = 2e^{5t} + 5e^{2t}, \quad y(0) = -3$

   ▶ **Solution.** This equation is linear with coefficient function $p(t) = -2$ so that an integrating factor is given by $\mu(t) = e^{\int(-2)\,dt} = e^{-2t}$. Multiplication of the differential equation by the integrating factor gives

   $$e^{-2t}y' - 2e^{-2t}y = 2e^{3t} + 5,$$

   and the left hand side is recognized (by choice of $\mu(t)$) as a perfect derivative:

   $$\frac{d}{dt}(e^{-2t}y) = 2e^{3t} + 5.$$

   Integration then gives

   $$e^{-2t}y = \frac{2}{3}e^{3t} + 5t + C,$$

   where C is an integration constant. Multiplying through by e^{2t} gives

   $$y = \frac{2}{3}e^{5t} + 5te^{2t} + Ce^{2t}.$$

   The initial condition $y(0) = -3$ implies $-3 = y(0) = 2/3 + C$ so that $C = -11/3$. Hence

   $$\boxed{y = \frac{2}{3}e^{5t} + 5te^{2t} - \frac{11}{3}e^{2t}.}$$

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(c) $y' - \dfrac{2}{t}y = 6t^4, \quad y(1) = -3$

▶ **Solution.** This equation is also linear with coefficient function $p(t) = -2/t$, so that an integrating factor is $\mu(t) = e^{\int (-2/t)\,dt} = e^{-2\ln t} = e^{\ln t^{-2}} = t^{-2}$. Multiplication of the differential equation by the integrating factor gives

$$t^{-2}y' - 2t^{-3}y = 6t^2,$$

and the left hand side is recognized (by choice of $\mu(t)$) as a perfect derivative:

$$\frac{d}{dt}(t^{-2}y) = 6t^2,$$

Integration then gives

$$t^{-2}y = 2t^3 + C,$$

where C is an integration constant. Multiplying through by t^2 gives

$$y = 2t^5 + Ct^2.$$

The initial condition $y(1) = -3$ implies $-3 = y(1) = 2 + C$ so that $C = -5$. Hence

$$\boxed{y = 2t^5 - 5t^2.}$$

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2. [18 Points]

(a) State Euler's formula for the complex exponential $e^{i\theta}$:

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

(b) Write the complex number $4e^{-i\pi/6}$ in rectangular form $x + iy$.

▶ **Solution.** From Euler's formula:

$$\begin{aligned} 4e^{-i\pi/6} &= 4(\cos(-\pi/6) + i\sin(-\pi/6)) \\ &= 4(\cos(\pi/6) - i\sin(\pi/6)) \\ &= 4(\sqrt{3}/2 - i/2) \\ &= \boxed{2\sqrt{3} - 2i.} \end{aligned}$$

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(c) Determine the polar expression $z = re^{i\theta}$ (i.e., find r and θ) for the complex number $z = 1 + \sqrt{3}i$.

▶ **Solution.** $r = |z| = \sqrt{1^2 + (\sqrt{3})^2} = 2$, and $\tan\theta = \sqrt{3}/1 = (\sqrt{3}/2)/(1/2)$, which we recognize from the table of common trigonometric values is $\tan(\pi/3)$. Hence

$$\boxed{z = 1 + \sqrt{3}i = 2e^{i\pi/3}.}$$

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3. [18 Points] Compute the Laplace transform of each of the following functions.

   (a) $f(t) = t^3 - e^{-9t} + 5$

$$\boxed{F(s) = \frac{6}{s^4} - \frac{1}{s+9} + \frac{5}{s}}$$

   (b) $g(t) = e^{-t}\cos 2t$

$$\boxed{G(s) = \frac{s+1}{(s+1)^2 + 4}.}$$

   (c) $h(t) = t^4 e^{-3t}$

$$\boxed{H(s) = \frac{24}{(s+3)^5}.}$$

4. [16 Points] A 400 gallon tank is initially full of brine which contains 60 pounds of salt. A solution containing 0.5 pounds of salt per gallon enters the tank at a rate of 6 gallons per minute. A drain is opened at the bottom of the tank through which the well stirred solution leaves the tank at the same rate of 6 gallons per minute. Let $y(t)$ denote the amount of salt (in pounds) which is in the tank at time t.

   (a) What is $y(0)$? $\boxed{y(0) = 60}$

   (b) Write the differential equation that $y(t)$ must satisfy.

▶ **Solution.** The balance equation

$$y'(t) = \text{rate in} - \text{rate out}.$$

The rate in is 0.5 lb/gal $\times$ 6 gal/min, i.e., 3 lb/min. The rate out is

$$\left(\frac{y(t)}{V(t)}\right) \times 6.$$

Since mixture is entering and leaving at the same volume rate of 6 gal/min, the volume of mixture in the tank is constant. Thus $V(t) = 400$. Hence $y(t)$ satisfies the equation

$$y' = 3 - \frac{6}{400}y,$$

so that the initial value problem satisfied by $y(t)$ is

$$\boxed{y' + \frac{3}{200}y = 3, \qquad y(0) = 60.}$$

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(c) Solve the differential equation to find $y(t)$.

▶ **Solution.** This is a linear differential equation with integrating factor $\mu(t) = e^{3t/200}$, so multiplication of the differential equation by $\mu(t)$ gives an equation

$$\frac{d}{dt}\left(e^{3t/200}y\right) = 3e^{3t/200}.$$

Integration of this equation gives

$$e^{3t/200}y = 200e^{3t/200} + C,$$

where C is an integration constant. Dividing by $e^{3t/200}$ gives

$$y = 200 + Ce^{-3t/200},$$

and the initial condition $y(0) = 60$ gives a value of $C = -140$. Hence, the amount of salt at time t is

$$\boxed{y(t) = 200 - 140e^{-3t/200}.}$$

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(d) How much salt is in the tank after 1 hour?

▶ **Solution.** This is obtained by taking $t = 60$ (minutes) in the previous equation:

$$\boxed{y(60) = 200 - 140e^{-180/200} \approx 143.08 \text{ lb}}$$

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Exam I Supplementary Sheet

A Short Table of Laplace Transforms

1. $\mathcal{L}\left\{af(t) + bg(t)\right\}(s) = aF(s) + bG(s)$

2. $\mathcal{L}\left\{e^{at}f(t)\right\}(s) = F(s-a)$

3. $\mathcal{L}\left\{t^n\right\}(s) = \dfrac{n!}{s^{n+1}}$

4. $\mathcal{L}\left\{e^{at}\right\}(s) = \dfrac{1}{s-a}$

5. $\mathcal{L}\left\{t^n e^{\alpha t}\right\}(s) = \dfrac{n!}{(s-\alpha)^{n+1}}$

6. $\mathcal{L}\left\{\cos bt\right\}(s) = \dfrac{s}{s^2+b^2}$

7. $\mathcal{L}\left\{\sin bt\right\}(s) = \dfrac{b}{s^2+b^2}$

8. $\mathcal{L}\left\{e^{at}\cos bt\right\}(s) = \dfrac{s-a}{(s-a)^2+b^2}$

9. $\mathcal{L}\left\{e^{at}\sin bt\right\}(s) = \dfrac{b}{(s-a)^2+b^2}$

Common trigonometric values

θ	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
$\cos\theta$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0
$\sin\theta$	0	$1/2$	$\sqrt{2}/2$	$\sqrt{3}/2$	1