Instructions. Answer each of the questions on your own paper. Put your name on each page of your paper. Be sure to show your work so that partial credit can be adequately assessed. *Credit will not be given for answers (even correct ones) without supporting work.* A copy of the table of Laplace transforms from the text will be supplied.

1. [20 Points] Compute the inverse Laplace transform of each of the following rational functions.

   (a) $F(s) = \dfrac{2s+5}{s^2+7s+10}$

   ▶ **Solution.** Use partial fraction decomposition to get

   $$F(s) = \frac{2s+5}{s^2+7s+10} = \frac{2s+5}{(s+5)(s+2)} = \frac{A}{s+5} + \frac{B}{s+2},$$

   where $A = \dfrac{2(-5)+5}{-5+2} = 5/3$ and $B = \dfrac{2(-2)+5}{-2+5} = 1/3$. Thus

   $$F(s) = \frac{5/3}{s+5} + \frac{1/3}{s+2},$$

   so that

   $$\boxed{f(t) = \frac{1}{3}e^{-2t} + \frac{5}{3}e^{-5t}.}$$

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   (b) $G(s) = \dfrac{s-5}{s^2-4s+20}$

   ▶ **Solution.** Complete the square in the denominator to get

   $$G(s) = \frac{s-5}{s^2-4s+20} = \frac{s-5}{(s-2)^2+4^2} = \frac{s-2}{(s-2)^2+4^2} - \frac{3}{(s-2)^2+4^2},$$

   so that formulas C.2.9 and C.2.10 give

   $$\boxed{g(t) = e^{2t}\cos 4t - \frac{3}{4}e^{2t}\sin 4t.}$$

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2. [20 Points] Use the Laplace transform method to find the solution $y(t)$ of the initial value problem

   $$y'' + 2y' + y = 4e^{-t}, \qquad y(0) = 0,\ y'(0) = 1.$$

▶ **Solution.** Let $Y(s) = \mathcal{L}\{y(t)\}$ be the Laplace transform of the solution function $y(t)$. Then applying the Laplace transform to the differential equation, and using formulas C.1.5 and C.1.6, we get

$$s^2Y(s) - sy(0) - y'(0) + 2(sY(s) - y(0)) + Y(s) = \mathcal{L}\left\{4e^{-t}\right\} = \frac{4}{s+1}.$$

Using the initial conditions $y(0) = 0$ and $y'(0) = 1$ we conclude

$$(s^2 + 2s + 1)Y(s) - 1 = \frac{4}{s+1},$$

or

$$(s+1)^2Y(s) = 1 + \frac{4}{s+1}.$$

Solving for $Y(s)$ gives

$$Y(s) = \frac{1}{(s+1)^2} + \frac{4}{(s+1)^3}.$$

Applying formula C.2.6 we conclude that

$$\boxed{y(t) = \mathcal{L}^{-1}\{Y(s)\} = te^{-t} + 2t^2e^{-t}.}$$

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3. [24 Points] Find the characteristic polynomial and the general solution of each of the following constant coefficient linear homogeneous differential equations:

(a) $3y'' - 5y' + 2y = 0$

▶ **Solution.** $p(s) = 3s^2 - 5s + 2 = (3s-2)(s-1)$, which has roots $2/3$ and 1. Hence

$$\boxed{y(t) = c_1e^{2t/3} + c_2e^t.}$$

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(b) $2y'' + 2y' + y = 0$

▶ **Solution.** $p(s) = 2s^2 + 2s + 1$ which has roots $\dfrac{-2 \pm \sqrt{4 - 4\cdot 2}}{4} = -\dfrac{1}{2} \pm \dfrac{1}{2}i$. Hence

$$\boxed{y(t) = c_1e^{-t/2}\cos(t/2) + c_2e^{-t/2}\sin(t/2).}$$

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(c) $y''' + 2y'' + y' = 0$

▶ **Solution.** $p(s) = s^3 + 2s^2 + s = s(s+1)^2$ which has roots $0, -1, -1$. Hence

$$\boxed{y(t) = c_1 + c_2e^{-t} + c_3te^{-t}.}$$

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4. [16 Points] Find the indicial polynomial and the general solution of each of the following Cauchy-Euler equations.

   (a) $2t^2y'' + 5ty' - 2y = 0$

   ▶ **Solution.** $q(s) = 2s(s-1) + 5s - 2 = 2s^2 + 3s - 2 = (2s-1)(s+2)$ which has roots $1/2$ and -2. Hence

$$\boxed{y(t) = c_1 |t|^{1/2} + c_2 t^{-2}.}$$

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   (b) $t^2y'' - 3ty' + 4y = 0$

   ▶ **Solution.** $q(s) = s(s-1) - 3s + 4 = s^2 - 4s + 4 = (s-2)^2$ which has a single root 2 of multiplicity 2. Hence

$$\boxed{y(t) = c_1 t^2 + c_2(\ln |t|)t^2.}$$

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5. [20 Points] Find the general solution of the following differential equation:

$$y'' + 6y' + 8y = 5e^{-2t}.$$

You may use any appropriate method.

▶ **Solution.** This equation can be solved by three different methods: Laplace transform, undetermined coefficients, and variation of parameters. We will use the method of undetermined coefficients.

The characteristic polynomial $p(s) = s^2 + 6s + 8 = (s+2)(s+4)$ so that the the associated homogeneous equation has general solution of the form $y_h = c_1e^{-2t} + c_2e^{-4t}$. $\mathcal{L}\left\{5e^{-2t}\right\} = \dfrac{5}{s+2}$ has denominator $v(s) = s + 2$, so that $p(s)v(s) = (s+2)^2(s+4)$. The relevant standard bases are then

$$\mathcal{B}_{p(s)v(s)} = \left\{e^{-2t},\, te^{-2t},\, e^{-4t}\right\} \quad \text{and} \quad \mathcal{B}_{p(s)} = \left\{e^{-2t},\, e^{-4t}\right\}.$$

The only term in the first basis that is not also in the second is te^{-2t}, so we conclude that a particular solution has the form $y_p = Ate^{-2t}$ where A is a constant to be determined by substitution into the original equation. Calculating derivatives gives

$$\begin{aligned} y_p' &= A(e^{-2t} - 2te^{-2t}), \text{ and} \\ y_p'' &= A(-4e^{-2t} + 4te^{-2t}). \end{aligned}$$

Substituting into the equation gives

$$y_p'' + 6y_p' + 8y_p = A(-4e^{-2t} + 4te^{-2t}) + 6A(e^{-2t} - 2te^{-2t}) + 8Ae^{-2t} = 2Ae^{-2t}.$$

Since this must equal $5e^{-2t}$ we conclude that $A = 5/2$, and hence $y_p = (5/2)te^{-2t}$. Then

$$\boxed{y_g = y_h + y_p = c_1e^{-2t} + c_2e^{-4t} + \frac{5}{2}te^{-2t}.}$$

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