Instructions. Answer each of the questions on your own paper. Put your name on each page of your paper. Be sure to show your work so that partial credit can be adequately assessed. *Credit will not be given for answers (even correct ones) without supporting work.* A copy of the Table of Laplace transforms from the text will be provided.

1. [**15 Points**] Find the unique solution of the initial value problem

$$y'' + 2y' + 10y = 0, \quad y(0) = 0, \quad y'(0) = 6.$$

Is the equation under damped or over damped? Does $y(t) = 0$ for some $t > 0$? If so, find the first $t > 0$ for which $y(t) = 0$. Carefully sketch the graph of your solution for the interval $0 \le t \le 2\pi$.

▶ **Solution.** This can be done more than one way. We will use the Laplace transform method. Let $Y(s) = \mathcal{L}\{y(t)\}$ be the Laplace transform of the solution function. Applying the Laplace transform to the differential equation gives

$$s^2Y(s) - sy(0) - y'(0) + 2sY(s) + 10Y(s) = 0.$$

Substituting the initial conditions $y(0) = 0$, $y'(0) = 6$ and solving for $Y(s)$ gives

$$Y(s) = \frac{6}{s^2 + 2s + 10} = \frac{6}{(s+1)^2 + 9}.$$

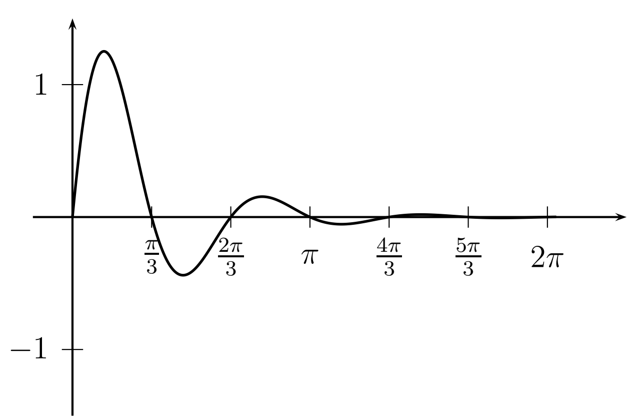
Taking the inverse Laplace transform (Formula C.2.9) gives

$$\boxed{y(t) = 2e^{-t}\sin 3t.}$$

This equation represents *underdamped* motion since the discriminant of the equation is $D = 2^2 - 40 = -36 < 0$. Since e^{-t} is never 0, it follows that

$$y(t) = 0 \iff \sin 3t = 0 \iff 3t = \pi k \iff t = \frac{\pi}{3}k$$

where k is any integer. Thus, the smallest positive t for which $y(t) = 0$ is $t = \pi/3$.



Graph of $y(t) = 2e^{-t}\sin 3t$.

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2. **[10 Points]** Find the Laplace transform of the following function:

$$f(t) = \begin{cases} 3t & \text{if } 0 \le t < 2, \\ 6 & \text{if } t \ge 2. \end{cases}$$

▶ **Solution.** First write $f(t)$ in terms of characteristic functions $\chi_{[a,b)}(t)$ and unit step functions $h(t-c)$:

$$\begin{aligned} f(t) &= 3t\chi_{[0,2)}(t) + 6h(t-2) \\ &= 3t(h(t) - h(t-2)) + 6h(t-2) \\ &= 3t + (6-3t)h(t-2). \end{aligned}$$

Now apply the second translation formula:

$$\begin{aligned} F(s) &= \frac{3}{s^2} + e^{-2s}\mathcal{L}\{6 - 3(t+2)\} \\ &= \frac{3}{s^2} + e^{-2s}\mathcal{L}\{-3t\} \\ &= \frac{3}{s^2} - e^{-2s}\frac{3}{s^2}. \end{aligned}$$

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3. **[10 Points]** Find the inverse Laplace transform of the following function:

$$F(s) = \frac{(s-2)}{s^2 - 4s + 13}e^{-\pi s/2}$$

▶ **Solution.** By formula C.2.10,

$$\mathcal{L}^{-1}\left\{\frac{(s-2)}{s^2-4s+13}\right\} = e^{2t}\cos 3t.$$

The second translation formula (C.1.4) then gives

$$\boxed{f(t) = (e^{2(t-\pi/2)}\cos 3(t-\pi/2))h(t-\pi/2).}$$

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4. **[15 Points]** Solve the following initial value problem:

$$y'' + 4y = 6\delta(t-\pi); \quad y(0) = 0,\ y'(0) = 0.$$

Give a careful sketch of the graph of the solution for the interval $0 \le t \le 2\pi$.

Recall that $\delta(t-c)$ refers to the Dirac delta function which produces a unit impulse at time $t = c$. In the Table of Laplace transforms, this is referred to as $\delta_c(t)$ (that is $\delta_c(t) = \delta(t-c)$). (See Formula 24, Page 431 of the Laplace Transform Tables).

▶ **Solution.** Let $Y(s) = \mathcal{L}\{y(t)\}$ be the Laplace transform of the solution function. Applying the Laplace transform to the differential equation gives

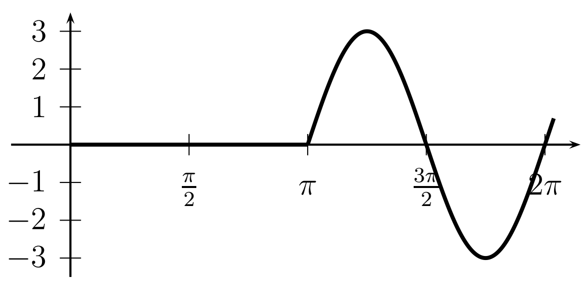
$$s^2Y(s) + 4Y(s) = 6\mathcal{L}\{\delta(t-\pi)\} = 6e^{-\pi s}.$$

Thus,

$$Y(s) = \frac{6}{s^2+4}e^{-\pi s}$$

and taking the inverse Laplace transform gives

$$\begin{aligned} y(t) &= (3\sin 2(t-\pi))h(t-\pi) \\ &= (3\sin 2t)h(t-\pi) \\ &= \begin{cases} 0 & \text{if } 0 \le t < \pi, \\ 3\sin 2t & \text{if } t \ge \pi. \end{cases} \end{aligned}$$



Graph of $y(t) = 2e^{-t}\sin 3t$.

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5. **[10 Points]** Let $A = \begin{bmatrix} x-2 & 1 & 0 \\ 4 & x+1 & 0 \\ 3 & -2 & x-4 \end{bmatrix}$.

   (a) Compute $\det A$.

   (b) For which values of x is A invertible?

▶ **Solution.** Using expansion along the third column gives

$$\begin{aligned} \det A &= (x-4)\det\begin{bmatrix} x-2 & 1 \\ 4 & x+1 \end{bmatrix} \\ &= (x-4)((x-2)(x+1)-4) \\ &= (x-4)(x^2-x-6) \\ &= (x-4)(x-3)(x+2). \end{aligned}$$

A is invertible if and only if $\det A \neq 0$. Since $\det A = 0$ if and only if $x = 4$, $x = 3$, or $x = -2$, it follows that A is invertible if and only if x is not 4, 3, or -2. ◀